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BRIDGING STUDENTS' IDEAS AND LESSONS' GOALS

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In order to be successful in using tasks in learning environments, it is necessary for teachers to use their students' solutions effectively. The purpose of this paper is to examine what teachers and students need to know in order to take advantage of students' solutions and learn new mathematical ideas. For this purpose, two teaching practices at elementary schools will be examined. This examination will show the importance of teachers' knowledge about various solutions and their relations as well as the important role which subproblems play during lessons.

INTRODUCTION

Problem solving approaches are helpful not for eliciting students' activities and motivating them, but also for providing students with opportunities to understand new mathematical ideas deeply (Schroeder & Lester, 1989; Nunokawa, 2005). When teachers pose problems in order to introduce new ideas, their students solve those problems using the mathematical knowledge the students have at that time. That is, students make sense of those problem situations with their mathematical knowledge at hand (Nunokawa, 2005). If their solutions and their discussions about solutions will lead to the new mathematical ideas, students can appreciate how the new mathematical ideas are related to the previous ideas and what kind of advantages the new ideas have over the previous ones.

This sounds a very simple story. But we should note here that there is an inherent difficulty in using problem solving approaches. In order to teach new mathematical ideas through problem solving, teachers need to highlight the students' solutions which are appropriate for teaching the new mathematical ideas. On the other hand, in order to involve their students in learning mathematics and make them motivated, teachers need to accept as various solutions as possible at least at the beginning of the class discussions.

In order to be successful in using tasks in learning environments, it is necessary for teachers to use their students' solutions effectively. Then, what do teachers and students need to know in order to take advantage of students' solutions? This paper will attempt to approach this "what" question through the examination of the teaching practices of two experienced elementary school teachers, especially focusing on how those teachers used their students' solutions to teach new mathematical ideas.

EXAMPLES 1: DIVISION OF FRACTIONS

The 6th graders in Japan learn the standard method of divisions whose divisors are fractions. An experienced teacher, Mr. Yuichi Takahashi, used the following problem in order to introduce this new division (Takahashi, 1992).

Problem 1: Juice is being poured into a tank. Six liter juice was poured during $\frac{2}{3}$ minute. How much juice will be poured during one minute?

The class explored this problem situation in the previous lesson and set up an expression $6 \div \frac{2}{3}$ to solve it. The goal of the second lesson was to find how to calculate this division. At the beginning of the lesson, the teacher asked his students to develop their own ways of calculating this division. After working individually, the teacher asked five students to present their ideas and the class began to examine those ideas. The five ideas presented by those students were as follow.

Student A: He drew the right number line and presented:

$$6 \div \frac{2}{3} = 6 \div 2 \times 3 = 9.$$

Student B: He presented: $6 \div \frac{2}{3} = 6 \times 3 \div 2 = 9.$

Student C: She presented: $6 \div \frac{2}{3} = (6 \times 3) \div \left(\frac{2}{3} \times 3\right) = 6 \times 3 \div 2 = 9.$

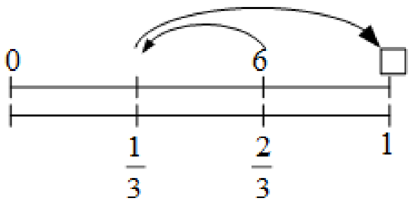

Figure 1: Student A's number line

Student D: He presented: $6 \div \frac{2}{3} = \left(6 \times \frac{3}{2}\right) \div \left(\frac{2}{3} \times \frac{3}{2}\right) = 6 \times \frac{3}{2} = 9.$

Student E: She presented: $6 \div \frac{2}{3} = 6 \times \frac{3}{2} = 9.$

Although the Student E's solutions is the standard method of division by fractions and the Student D's solutions is more directly related to it than other solutions, the teacher asked Student A to present his solution first and Student B and C to follow him.

When examining their solutions, Student C and D expressed a doubt about the A's solution. They asked the supporter of Student A why his arithmetic expression included a multiplication even though the original expression did not have a multiplication. They pointed out that $6 \div \frac{2}{3}$ has only a division and $\frac{2}{3}$ has no multiplication ($\frac{2}{3} = 2 \div 3$). Student A and his supporter explained that a multiplication was necessary for finding the amount of juice poured during one minute on the basis of the amount poured during $\frac{1}{3}$ minute in Fig 1. While Student C and D accepted the number line Student A constructed, they continued to insist that it was strange that a multiplication appeared in the A's expression and this expression, $6 \div 2 \times 3$, did not reflect the original division.

Of course, the expressions used by Student C and D also included multiplications. But the students did not refuse the C's and D's solutions possibly because they had learned the properties of division as follows: $a \div b = (a \times c) \div (b \times c)$ and $a \div b = (a \div c) \div (b \div c)$. The solutions of Student B could be also accepted referring to the Student C's solution.

The Teacher's Acceptance of the Students' Criticism

Mr. Takahashi reported that he got confused here as he did not expect such a discussion in advance. Seeing the students' solutions again, the teacher decided to utilize the number line introduced by Student A in order to help the students examine their solutions on equal footing. The teacher required his students to re-present the solutions of Student B, C, and D using number lines. It can be said that the teacher set a subproblme here as follows: (Subproblem 1) What is the similarities and the differences between Student A's solution and the other students' solutions.

The students represented the C's solution using the number line in Fig. 2. The right arrows towards the gray box and "2" may represent multiplication by 3. The students called the stopover (i.e. the gray box and the "2" below it) "a base," and explained that this idea changed the divisor into an integer, 2, by making the base so that they could circumvent dividing 6 by a fraction directly.

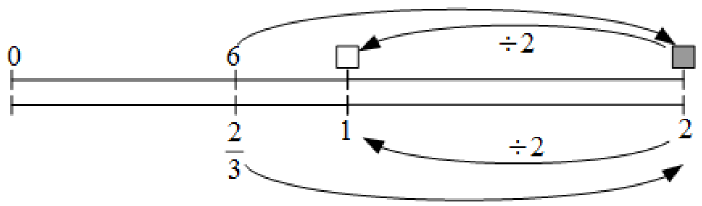

Figure 2: The number line representing C's solution

Then, the teacher asked the student to compare the solutions of Student A and C. The students began to explain the difference and the similarity between these solutions referring to the two number lines (Fig. 1 and 2) and using the term "a base" as follows: "The ways of making bases are different. Student B and C made a base at an integer (i.e. 2) to change the divisor into an integer. Student A made his base at one third." (See Fig. 3); "All the students made their bases first, and then, they try to find the amount poured during one minute because we cannot divide it by two thirds."

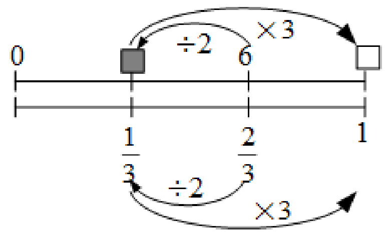

Figure 3: Elaborated version of A's number line

The teacher summarized these explanations and validated the use of a multiplication as follows: "Because we cannot divide 6 by a fraction directly, we need to take two steps to implement the computation. First, we try to find a base at an integer or a unit fraction. Then, we find the quantity per one minute. Two movements appeared on the number line. The first one is a movement for finding a base, and the second one is a movement for finding the quantity per one minute. One of these movements required a multiplication."

After this explanation, the teacher encouraged his students to compare the Student D's solution to the solutions of Student A and C. The D's solution can be represented using the number line like Fig. 4. Comparing the solutions of Student D and of Student A and C, the class characterized the D's solution as the way without constructing a base.

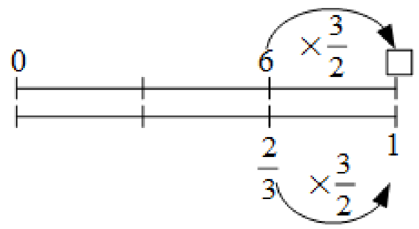

Figure 4: The number line of D's solution

The D's solution easily resulted in the E's solution, the standard method of division by fractions. This characterization of the D's solution, "the way without constructing a base," might provide the students with the reason why the divisor fraction, $\frac{2}{3}$, should be inverted and the multiplication should be used. The students' comparison of their ideas led to the new idea to be introduced in this lesson and made them understand the validity of this target idea.

Implication of Example 1

First, this example suggests that teachers need to know various possible solutions to the tasks and the relationships between those solutions and the target ideas. While the D's solution illustrated the foundation of the standard method shown in the E's solution, the D's and C's solutions have the common idea behind them, the property of divisions, $a \div b = (a \times c) \div (b \times c)$. The C's and A's solutions could be related to each other by referring to proportional reasoning and the idea of "base." Because the teacher knew such relationships between possible solutions, he could anticipate how to relate those solutions to achieve the goal of this lesson. Second, teachers need to know the representations which can be clues for students' discussions. In this example, the teacher adopted number lines, which highlighted the similarity and the difference between the solutions and enabled the students to relate their solutions. It should be noted here that in order to choose number lines as an appropriate representation in this context, the teacher needed to know that all of the solutions the student presented in the lesson were backed by proportional reasoning.

In order to follow this lesson, first, the students needed to know the above-mentioned property of divisions. If they did not know this property, they could not understand the C's and D's ideas and then could not understand the reason why the dividend should be multiplied by the inverse of the divisor. They also needed to know how to represent this property of division or proportional reasoning on a number line (Nunokawa, 2012, 2015). Second, the students needed to know the meaning of division in this problem situation in order to approach the problem in a meaningful way: the division provides the amount per unit. In all of the number lines, the arrows on the lower lines start from $\frac{2}{3}$ and end at 1. The above meaning of division made it easy to understand the relation between the division and the number lines. Third, the students needed to know some socio-mathematical norms of mathematics lesson (Cobb & Yackel, 1998). If they did not require the reason why the divisor should be inverted and was satisfied with the memorization of the standard method, the students were unwilling to participate in such a discussion and examine the peers' solutions.

EXAMPLES 2: QUANTITY PER UNIT

The 5th graders in Japan learn the idea of a quantity per unit. Another experienced teacher, Mr. Masataka Mageshi, used the following problem, a kind of ratio comparison problem (Lamon, 2007), in order to introduce this idea to his 5th grade students (Mageshi, 1997).

Problem 2: You are planning a room assignment for our school trip. The table on the right shows the sizes of three rooms and the numbers of students who will use these rooms. Which room will be most crowded?

	Room Size (# of Tatami Mats)	# of Students
Room A	8 tatamis	6 students
Room B	10 tatamis	6 students
Room C	10 tatamis	8 students

Here, Tatami is a Japanese traditional flooring-mat whose size is 90 cm×180 cm. Numbers of tatamis are often used to represent sizes of rooms in Japan.

The teacher first asked the students whether they could determine the least crowded room. The students answered at once that Room B is the least crowded because "Although the same number of students will use Room A and B, Room B has more tatamis than Room A" and "Although Room B and C have the same number of tatamis, the number of students who will use Room B is smaller than that of Room C." Furthermore, Student K told the following idea: "When each student will use one tatami, Room A and C will have two extra tatamis [8-6=2 and 10-8=2] and Room B will have four extra tatamis [10-6=4]. As Room B will have most extra tatamis, it is the least crowded." K's idea was based on the additive strategy (e.g. Misailidou & Williams, 2002).

This K's opinion was met by a mixed reaction. Some students partly opposed Student K as follows: "In this case, we could conclude that Room B is the least crowded by finding out the differences between the numbers of tatamis and of students. But we cannot always compare the crowdedness of rooms on the basis of those differences." Here a subproblem seemed to be set through the initial discussion: (Subproblem 2) Can we compare the crowdedness of Room A and C using the differences between the numbers of tatamis and of students, or should we use other methods? This discussion also clarified that rooms can be easily compared when one of the factors, the number of tatamis or that of students, is the same, like Room A vs. B and Room B vs. C. This led to the necessity of the new idea, a quantity per unit: It is very useful for comparing things which have no common number in both factors.

After the students worked individually, the class discussion started again. In solving such a task, the following idea is usually developed, which enables students to compare two rooms without resorting to the quantities per unit: When changing the numbers of students to their LCM by using proportional reasoning, Room A has 24 students and 32 tatamis and Room C has 24 students and 30 tatamis and, as Room C has less tatamis, it is more crowded than Room A. The idea of a quantity per unit is closely related to this idea: changing the numbers of students into "1," instead of LCM, by proportional reasoning (Fig. 5). Therefore, the teacher might be able to relate the students' solution to the target idea and introduce this idea as a variation of their solution.

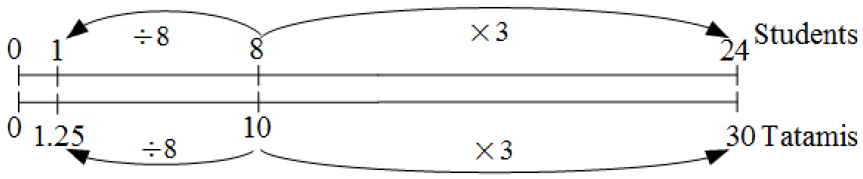

Figure 5: Proportional reasoning and the quantity per unit

According to Mageshi (1997), there were still some students who supported the use of the differences. There were also many students who could not determine confidently whether the use of differences was inappropriate or not in this case. As the above student K pointed out, it seems natural to assign one tatami to each student and check the numbers of extra tatamis (Figure 6).

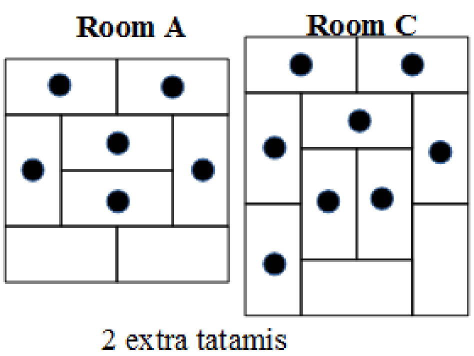

Figure 6: Assignment of tatamis to students

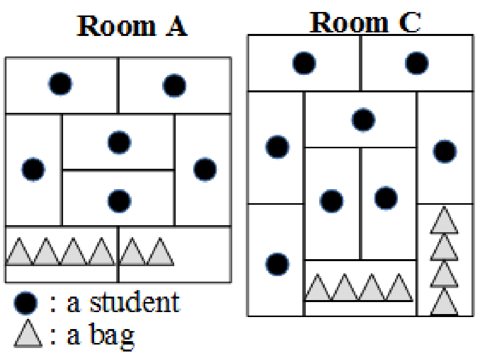

Figure 7: Modification of the Assignment

Finally, some students developed an opposition to this additive-strategy-based solution. They insisted that even though both rooms have 2 extra tatamis, 8 students in Room C must share those two tatamis to keep their baggage while only 6 students in Room A can use extra two tatamis (Fig. 7). When a quarter tatami will be used to keep one bag, Room A can still have a half tatami as extra space. This implies that Room A is less crowded than Room C.

This opposition showed not only the limitation of the additive-strategy-based solution, but also the possible link between this solution and the idea of a quantity per unit. When taking account of the numbers of tatamis which each student can use to keep his bag and assigning extra tatamis to students, each student in Room A can use about 1.33 tatamis ($8 \div 6$) while each student in Room C can use only 1.25 tatamis ($10 \div 8$). This is the same as finding the numbers of tatamis per student. The students noticed this link, produced the idea of a quantity per unit from this line of thought, and understood the validity of the use of a quantity per unit in this situation.

Moreover, their examination of the additive-strategy-based solution as well as the opposition to it made explicit the implicit assumption behind the idea of a quantity per unit: All tatamis in the room should be assigned evenly and fairly, and any extra space cannot be admitted. In this sense, this examination was helpful for students to understand the idea of a quantity per unit more deeply.

Implication of Example 2

First, teachers need to know various possible solutions to the tasks and the relationships between those solutions and the target ideas. In the above problem, the solution based on LCM and the target idea, a quantity per unit, have the common idea behind them: Changing the numbers of students into the same number by using proportional reasoning. Second, teachers need to know typical misconceptions related to the problems and the relationships between those misconceptions and the target ideas. In this example, assigning extra tatamis to students could bridge the gap between this misconception and the target idea.

In order that they could follow the above lesson, there was something the students in his class needed to know in advance. First, the students needed to know proportional relationships between the numbers of tatamis and of students: To keep the crowdedness of a room, the numbers of tatamis and of students should be multiplied by the same number. Second, in order to approach the problem in a meaningful way, the students needed to know the feeling of crowdedness. Mr. Mageshi provided them with an opportunity to experience the sizes of rooms and their crowdedness before solving the problem. He made the sizes of 8- and 10-tatami rooms with tapes on the floor. The students lay down on these spaces and felt the crowdedness of these rooms. This experience might support the students' thinking during their problem solving.

Third, the students needed to know some social and socio-mathematical norms of mathematics lesson. When some students claimed that Room C is more crowded by seeing the drawings they constructed, other students opposed them because “seeing the drawings” cannot validate their decision. This is a manifestation of a socio-mathematical norm about explanations acceptable in this mathematics class.

FACTORS SUPPORTING SUCCESSFUL USE OF TASKS

The above examinations of the two examples suggested that teachers need to know various solutions and how those various solutions can lead to the target mathematical ideas. This requires teachers to know the target mathematical ideas well and to know students' strategies and misconceptions (Misailidou & Williams, 2002). And, if possible, teachers should construct or arrange the tasks by taking account of what kinds of solutions can be anticipated and how those anticipated solutions can lead to the mathematical ideas to be introduced.

It should be noted here that what students need to know is what teachers need to develop in their students in the long-term. The teachers' action of encouraging students to relate and examine their various solutions and showing how their solutions can be related to the target mathematical ideas will be reflected in students' dispositions and will develop certain social and socio-mathematical norms in their classes, which will support future problem solving and discussions in their classes.

Some mathematical ideas which are necessary for problem solving and discussions must also be developed in students in the long-term. Unless proportional reasoning, for example, has been a part of classroom mathematical practice (Cobb & Yackel, 1998), students can hardly develop solutions based on it and examine their solutions by resorting to it. It is necessary for teachers to develop classroom mathematical practices enough for solving the problems and discussing anticipated solutions before teachers use the problems in order to introduce new mathematical ideas.

Another point we should note is the fact that both of the two teachers seemed to set subproblems after the initial class discussion. In Mr. Takahashi's lesson, the students had discussed the validity of Student A's solution just before the teacher set the subproblem. After this stage of the lesson, the students represented their solutions on number lines and explored the similarity and the difference between those solutions. The teacher asked how to calculate the division by a fraction at the beginning of the lesson, but the subproblem put more emphasis on the students' reflection of the procedures of division rather than on finding the answer and might shift focus of the class discussion to relations of the procedures.

In Mr. Mageshi's lesson, the students had discussed the additive-strategy-based solution just before the teacher set the subproblem. After this stage of the lesson, the students focused on Room A and C and explored how they could compare these two rooms rather than simply tried to find the most crowded room. In other words, the subproblem might shift focus of the students' attention from the answer of the original problem to the procedures for finding that answer.

The use of appropriate problems is important to achieve the goals of lessons. But it seems also important to set appropriate subproblems during the lessons. These subproblems may be set on the basis of the class discussions and shift focus of the class discussions to the reflection or the examination of procedures which students initially used to solve the original problems. As the reflection and the examination of procedures is a key in formulating mathematical ideas (e.g. Gravemeijer, 1997), setting appropriate subproblems during the lessons is a key to the effective use of problems in order to introduce mathematical ideas. Therefore, it seems useful for teachers to know what kinds of subprolbmes can be set from their original problems and what kinds of initial discussion can lead to those subproblems. In a sense, these subproblems are the "real" problems which students have to think and discuss in the lessons in order to understand the new mathematical ideas well.

CONCLUDING REMARKS

To summarize the above discussion, what teachers need to know about tasks in order to be successful in using them is the knowledge about various solutions of the tasks "to anticipate student responses to questions and tasks" (Fujii, 2015, p. 45) and to design paths to the goals of lessons, and what students need to know is the knowledge which reflects the classroom mathematical practice and socio-mathematical norms in their classrooms. Teachers need to develop appropriate classroom mathematical practice and socio-mathematical norms in advance. We should also note the important role which subproblems may play as stepping-stones between students' various solutions and target mathematical ideas.

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