ICME-10, TSG-18

MATHEMATICAL PROBLEM SOLVING AND LEARNING

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The issues that the organizers advised me to discuss in this manuscript are;

(a) What sort of learning do we expect students to obtain from problem solving experiences?

(b) In order to maximize this learning, what kind of experiences should the students be exposed to, and with what pedagogical support?

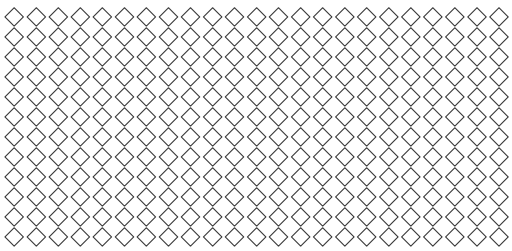
(c) Should this learning be aimed merely towards enhancing abilities in general problem solving, or should we use the problem solving perspective as a way also to treat mathematical theories?

After introducing a diagram representing (simplified) conception of mathematical problem solving, I will treat question (a) and (b) in Section 2 using that diagram. Extending the discussion about one of the types of problem solving identified in Section 2, question (c) will be discussed in Section 3.

1. Two Examples: Basic Standpoint of This Manuscript

Example 1 (Introduction problem of 2-digit numbers multiplication from Japanese 3rd grade textbook)

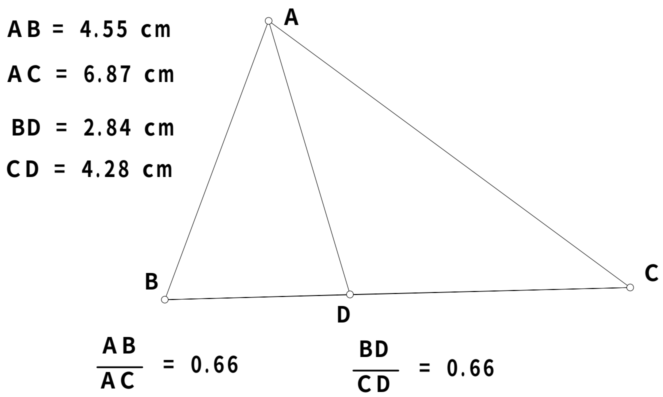
We will make small towers using plastic blocks. We will hand out 23 blocks to each child. How many blocks do we need if there are 12 children?



Adding the number of blocks, 23, twelve times or counting the total number of blocks. Or exploring the situation so that students can associate their knowledge with it.

Example 2 (a problem from a Japanese 9th grade textbook)

Assume $\angle BAD = \angle CAD$ in the following situation.



Prove that AB:AC=BD:CD.

Exploring the situation (modifying diagrams, using dynamic geometry software etc.) so that students can associate their knowledge with it.

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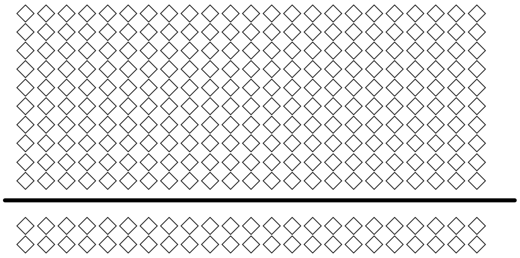
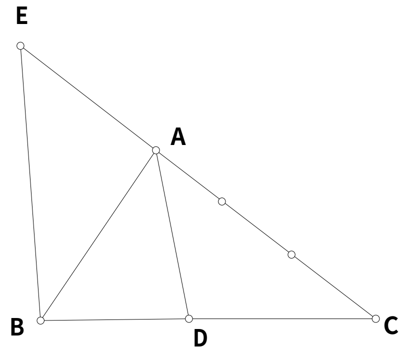
 If the number of children is split into groups of 10 and 2, the number of blocks for those 10 children can be found by multiplying 23 by 10, and the number of blocks for 2 children can be found by multiplying 23 by 2. Then, the total number can be calculated by adding those two products. There are also some children using other ways of decomposing 12 (e.g. 6+6). Here we find new information (total number is 276, or it is the sum of the numbers for 10 children and the number for the rest) about the above situation.	 Take the point E on the extension of CA so that AE=AB. Then it can be shown that ΔCAD is similar to ΔCEB and BD:CD=AE:AC. Here we find new information (AB:AC=BD:CD, or similar triangles can be constructed) about the above situation (a triangle with a bisector of one angle).

What sort of learning do we expect our students to obtain from above problem solving experiences? The answer to this question seems to vary depending on which aspect of the above solving processes is emphasized.

For example:

* If we hope that our students pay enough attention to the new information (i.e. a certain way of decomposing 12, AB:AC=BD:CD in the above situation[1)]), what we expect them to obtain is newly constructed mathematical knowledge.
* If we hope that in the above situations, our students can appropriately use knowledge they learned before, what we expect them to obtain is an experience of applying that knowledge (i.e. multiplication of 2-digit number and 1-digit number, a theorem about similar triangles).

This examination of the above examples shows that answers to question (a) vary depending on our intention in our using a particular problem solving experience at a particular teaching/learning situation.

For investigating some types of learning experienced through mathematical problem solving, here I would like to introduce my conception of problem solving processes and a diagram representing it.

In this manuscript, the image or conception of mathematical problem solving is as follow:

Mathematical Problem Solving is a thinking process in which a solver tries to make sense of a problem situation using mathematical knowledge he/she has and attempts to obtain new information about that situation and/or to "resolve the tension or ambiguity" (Lester & Kehle, 2003) about that.

For example:

*In Example 1, students try to make sense of the problem situation using their knowledge about multiplication of a 2-digit number and a 1-digit number or multiplication of a 2-digit number and a multiple of 10. They obtain new information about the total number of blocks or ways of arranging objects in the situation so that they can manage those objects using only existing knowledge.

*In Example 2, students try to make sense of the problem situation using their knowledge about triangles or similar figures. They obtain new information about the ratio of the sides or the mechanism (Nunokawa & Fukuzawa, 2002) making that ratio possible.

This image can be represented like Figure 1 [2].

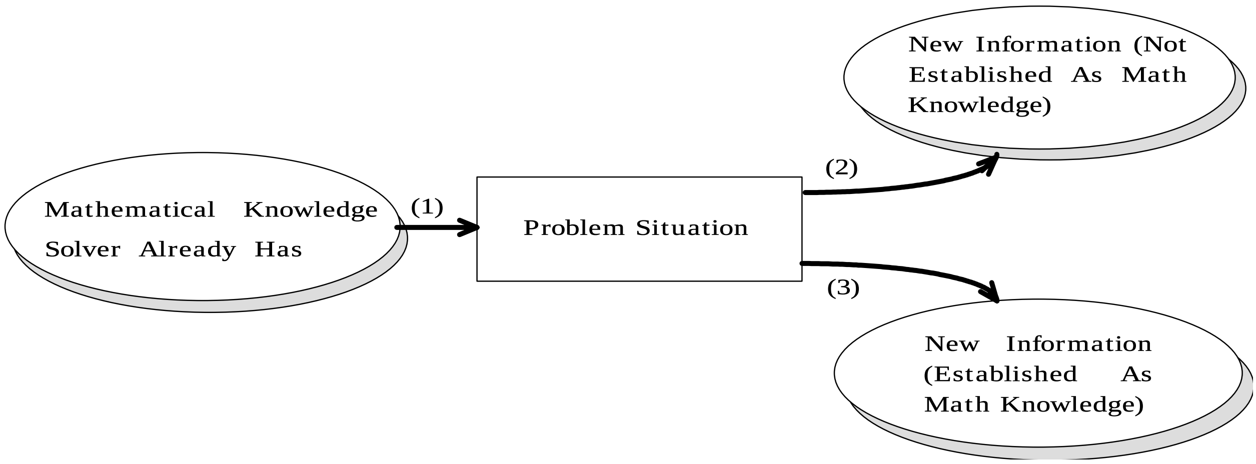


Figure 1. An image of problem solving processes

This model shows: Solvers explore a problem situation with his/her mathematical knowledge and try to obtain new information about the situation.

2. What We Expect Students to Learn in Problem Solving and How We Can Support Them

Depending on which part in the above diagram we emphasize, we can expediently identify some experiences which we expect our students to have in solving problems (Table 1)[3].

Table 1: Some experiences students can have in their problem solving processes

emphasis	Process Itself		Knowledge the Solver Already Has	New Information
(1) => (2)	Word- or Process-Problem Solving Mathematical Modeling	(teaching *about* problem solving)	Application of Acquired Knowledge (teaching *for* problem solving)	Exploration and Understanding of the Situation
(1) => (3)	Doing Mathematics			Construction of Mathematical Knowledge (teaching *via* problem solving)

(a) Emphasizing the process itself: (1)=>(2) or (1)=>(3)

This corresponds to "teaching *about* problem solving" in Schroeder & Lester (1989).

What can be learned: How to carry on solving processes or put forward thinking; ways of mathematical modeling or critical competence (Blum & Niss, 1991); ways of doing mathematics (Schoenfeld, 1994) or picture of mathematics (Blum & Niss, 1991).

Possible support:

* Developing students' repertories of ways of probing problem situations (Nunokawa, 2000; Stylianou, 2002) or generating pieces of information about situations (Lawson & Chinnappan, 1994), including heuristic strategies.
* Providing tools for probing situations (e.g. dynamic geometry software (Nunokawa & Fukuzawa, 2002)).
* Developing their meta-cognitive competences (Schoenfeld, 1992), through, for example, scaffolding (Holton & Thomas, 2001) or cooperative learning (Artzt, 1996).
* Growing appropriate beliefs in students (Schoenfeld, 1992).
* Asking questions which we expect students to use in future problem solving (Polya, 1945/1973).
* Inviting students to participate in practices of doing mathematics (Schoenfeld, 1994); cognitive apprenticeship (Brown *et al.*, 1989) about mathematical problem solving or doing mathematics.

(b) Emphasizing the application of mathematical knowledge students have acquired: (1)

This corresponds to "teaching *for* problem solving" in Schroeder & Lester (1989).

What can be learned: How and when to apply mathematical knowledge students have.

Possible support:

* Enriching students' schemata of typical situations (Greeno, 1987; Verschaffel & De Corte, 1997):

Greer (1992) recommended that "students need to learn about a much wider range of situations modeled by the operations."

* Enriching domain-specific schemata (Owen & Sweller, 1985) in which targeted mathematical knowledge is inherent: Owen & Swellter (1985) recommended providing problems with the specificity of their goal reduced because working with those problems facilitated acquisition of schemata.
* Changing situations into ones which can be easily associated to targeted knowledge (Hudson, 1983); using situations familiar or meaningful to students (Bottge *et al*., 2001; Woolnough, 2000).
* Directing students' attention to appropriate interpretation of solutions (Silver & Shapiro, 1992).
* Reducing math anxiety (Archambeault, 1993).

(c) Emphasizing new information about a problem situation which will not be established as mathematical knowledge: (2)

What are important in this type are pieces of new information about explored situations. Mathematical modeling in cross curricula is included in this type.

What can be learned: New information about a explored situation; new conception of mathematics (e.g. power of mathematics).

Possible support:

* Selecting situations which students are interested in and about which they want to know new information.
* Providing situations where the power of mathematics can be illuminated (e.g. the situations where unanticipated results will be obtained through mathematical analysis (Nunokawa, 2001)).
* Providing tools which help students investigate the situations (e.g. graphic calculators (Osawa, 1996), computers). If necessary, teachers introduce appropriate mathematical knowledge to compensate the shortage of students' knowledge (Matumiya & Yanagimoto, 1995).

(d) Emphasizing new information which will be established as mathematical knowledge: (3)

This corresponds to "teaching *via* problem solving" in Schroeder & Lester (1989)[4)].

What can be learned: New mathematical knowledge, which is supported by other mathematical knowledge and/or knowledge and representations in other fields (e.g. counters or blocks for number operations, falling body for quadratic functions etc); "a composite understanding" based on situated cognition (Brown *et al*., 1989).

Possible support:

* Providing pedagogical objects or tools which can bridge knowledge solvers have and new knowledge to be constructed (e.g. base-10 blocks); Providing "surrounding support that together promote correct methods" (Fuson & Burghardt, 2003).

* Eliciting mathematical knowledge from students' results of problem solving and formulating it; facilitating students reflecting their own activities (De Corte *et al*., 1996).
* Supporting students so that their ideas become more mathematical (e.g. from description of situations to mathematical notations (Fuson & Burghardt, 2003), from model-of to model-for (Gravemeijer, 1997)).
* When using measured data, it may be necessary to make a context where a certain way of seeing the data can be dominated (Christiansen, 1997).
* Creating classroom norms where constructions of knowledge are central (Carpenter *et al*., 1999); developing socio-math norm (Cobb & Yackel, 1998) about what kinds of knowledge are respected in mathematics and when knowledge is mathematically validated.

The above analysis implies that we should pay enough attention to features of a problem situation we use so that the situation can elicit experiences we expect our students to have.

For type (a) learning: Can the situation become an authentic problem situation for students?

For type (b) learning: Can the situation appropriately elicit students' mathematical knowledge we expect them to use?

For type (c) learning: Can the situation produce interesting results when students apply their mathematical knowledge?

For type (d) learning: Can the situation produce results leading to targeted mathematical knowledge when students apply their mathematical knowledge?

3. Should We Use the Problem Solving Perspective as a Way to Treat Mathematical Theories?

If we follow the discussion about type (d) learning in the previous section, the use of the problem solving perspective in teaching mathematics can be supported because of the following reasons.

(i) It makes it possible that new knowledge students need to learn is connected with mathematical or non-mathematical knowledge they already have. These connections may constitute students' understanding and support their "adaptive expertise" (Baroody, 2003), especially when students construct those connections by themselves.

(ii) It makes students' learning of targeted mathematical knowledge compatible with the recent view that knowledge is situated (Brown *et al*., 1989; De Corte *et al*, 1996). In such learning, it may be illuminated what kinds of roles targeted mathematical knowledge can play in the situations, how valuable it is, and why it is necessary.

(iii) It demonstrates the picture of mathematics as human activities (De Corte *et al*, 1996) and facilitates students becoming autonomous learners.

Possible supports for type (d) learning also show, however, that there are some limitations or difficulties in such teaching/learning approach as follows:

* Students' thinking is sometimes needed to be guided by a teacher so that targeted mathematical knowledge can be formulated (Fuson & Burghardt, 2003).
* When various solutions are presented by students, it is not always easy to integrate them and make them converge toward mathematical ideas. We need a context where mathematically canonical ideas are naturally selected by learners (Nunokawa & Kuwayama, in press).

For example:

* Assume that we are attempting to introduce irrational numbers using problems where students are asked to make squares whose areas are 2, 4, 5, and 9. This situation may help students recognize that there are quantities which correspond to the lengths of sides of those squares. How can it happen, however, that students spontaneously insist that there should be new types of numbers which can express those quantities?
* Assume that students have learned the solution of the equation in the form of $(x\text{-}a)^2=b$. Can we make a problem situation where students can construct solution methods of the equation $ax^2+bx+c=0$ which can lead to the solution formula of quadratic equations? Some Japanese mathematic teachers expect that their 9th grade students will find the basic idea supporting this formula in the situation where they can operate the tiles shown in Figure 2. But even the explanation of the formula using these tiles is not easy to understand for 9th grade students (e.g. Why do we need to add an extra tile?).

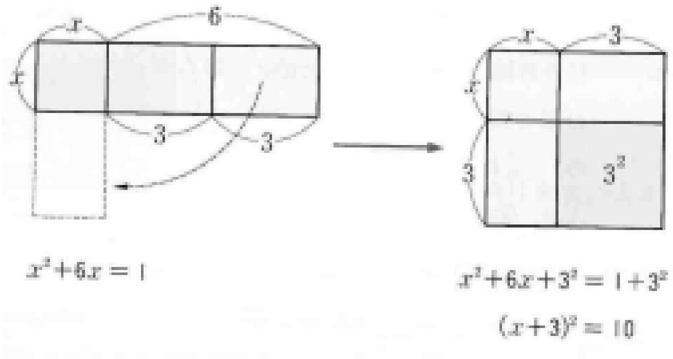


Figure 2 Tile model for quadratic equations in a Japanese textbook

These examples imply that there may be some parts of a mathematical theory which students need a great deal of teachers' intervention to learn. In such cases, we need to examine whether students can gain the above-mentioned benefits through that problem solving even when we provide such intervention or guide students' thinking processes.

On the other hand, those examples also suggest that if students cannot resolve those problem situations using knowledge students have at that time (*i.e.* failure in type (b) learning), this failure would demonstrate values and functions of the targeted topic or knowledge (e.g. why do we need such a new concept? How can we resolve such a complex situation?). Type (c) learning implies another way of illuminating values and functions of the targeted topic or knowledge: After learning that knowledge, students try to solve a problem situation using that knowledge and experience that it can induce information about the situation which is difficult to obtain without that mathematical knowledge.

The analysis in this section seems to indicate that instead of asking whether we should teach a whole mathematical theory through problem-solving activities, we should ask this issue as follows:

Reformulation of the question: What aspects of a targeted mathematical theory should we treat through problem solving? Concerning that theory, what kinds of experiences do we want to provide our students through problem solving?

The latter question can be restated as the following three questions, using the framework presented in the previous section:

* Where do we need type (d) learning?
* Where do we need type (c) learning?
* Where do we need type (b) learning?

Note

1) In fact, there are junior high school mathematics teachers in Japan who hope that their students memorize this ratio and use it in solving entrance examinations.

2) Of course, this diagram oversimplifies problem solving processes. Problem solving processes usually include representing situations, exploring situations so that existing mathematical knowledge can be applied, and exploring situations taking into account of newly found information (e.g. Lester & Kehle, 2003; Nunokawa, 1994, 1995). This simplification was made in order to focus on relationships between problem solving processes and mathematical knowledge, which is one of the important issues of this manuscript.

3) Many cases of use of problem solving experiences may not include only one of the four types, but include two or more of them simultaneously (see, for example, the model-eliciting activities (Lesh & Doerr, 2003; Lesh & Harel, 2003). This categorization was made so that the author could be more sensible to what we expect students to obtain from problem solving experiences and how it can be experienced by students.

4) Tyep (d) learning also seems to correspond to Baroody's (2003) "the investigative approach." He characterized it as follows (p. 22): (a) Like the conceptual approach, an aim of mathematics instruction is to help students learn needed facts, rules, formulas, and procedures in a meaningful fashion; (b) Like the problem-solving approach, students are regularly engaged in mathematical inquiry; (c) Like the problem solving approach, a teacher indirectly incites doubt, curiosity, or cognitive conflict by posing worthwhile tasks and by creating a social environment that encourage questioning, inquiry, and reflection; (d) Unlike the problem-solving approach, children's active construction of understanding is mediated, guided, and prompted by the teacher.

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