

MAASS RELATIONS FOR GENERALIZED COHEN-EISENSTEIN SERIES OF DEGREE TWO AND OF DEGREE THREE

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ABSTRACT. The aim of this paper is to generalize the Maass relation for generalized Cohen-Eisenstein series of degree two and of degree three. Here the generalized Cohen-Eisenstein series are certain Siegel modular forms of half-integral weight, and generalized Maass relations are certain relations among Fourier-Jacobi coefficients of them.

1. INTRODUCTION

1.1. The generalized Cohen-Eisenstein series are certain Siegel modular forms of half-integral weight, which have been introduced by Arakawa [Ar 98]. Originally the Cohen-Eisenstein series have been introduced by Cohen [Co 75] as one variable functions. In the case of degree one, it is known that the Cohen-Eisenstein series correspond to the Eisenstein series with respect to $SL(2, \mathbb{Z})$ by the Shimura correspondence. On the other hand, in arbitrary degree we can identify the generalized Cohen-Eisenstein series with Jacobi-Eisenstein series of index 1. Hence, we can expect some significant properties on the generalized Cohen-Eisenstein series.

This article is devoted to show generalized Maass relations for generalized Cohen-Eisenstein series of degree two (Theorem 1.1) and of degree three (Theorem 1.4). Here, the generalized Maass relations are certain relations among Fourier coefficients, and these relations are equivalent to certain relations among Fourier-Jacobi coefficients.

It is known that a certain kind of Siegel modular forms of degree two are obtained from elliptic modular forms through the Saito-Kurokawa lift. Such Siegel modular forms are characterized by the Maass relation. The generalized Maass relations for generalized Cohen-Eisenstein series of degree three (Theorem 1.4) are applied for the images of the Duke-Imamoglu-Ikeda lift. It means that certain Siegel cusp forms of half-integral weight of degree three satisfy generalized Maass relations. These relations give a key of the proof for a certain lifting which is a lifting from pairs of two elliptic modular forms to Siegel modular forms of half-integral weight of degree two (cf. [H 11].) This lifting has been conjectured in [H-I 05].

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Because generalized Maass relations in this paper are expressed as relations among Fourier-Jacobi coefficients, we shall explain the Fourier-Jacobi coefficients of Siegel modular forms. Let F be a Siegel modular form of degree n of integral weight or half-integral weight, which is a holomorphic function on the Siegel upper half space $\mathfrak{H}_n$ of size n . We consider an expansion:

$$F\left(\begin{pmatrix} \tau_{n-m} & z \\ t z & \omega_m \end{pmatrix}\right) = \sum_{S \in \text{Sym}_m^*} \phi_S(\tau_{n-m}, z) e^{2\pi\sqrt{-1}\text{tr}(S\omega_m)},$$

where Sym_m^* denotes the set of all half-integral symmetric matrices of size m , and where $\begin{pmatrix} \tau_{n-m} & z \\ t z & \omega_m \end{pmatrix} \in \mathfrak{H}_n$, $\tau_{n-m} \in \mathfrak{H}_{n-m}$, $\omega_m \in \mathfrak{H}_m$ and $z \in M_{n-m,m}(\mathbb{C})$. Then ϕ_S is a Jacobi form of index S of degree $n - m$. The above expansion is called the *Fourier-Jacobi expansion* of F . Moreover, the above forms ϕ_S are called *Fourier-Jacobi coefficients* of F . The generalized Maass relations are certain relations among ϕ_S , and such relations are equivalent to certain relations among Fourier coefficients of F .

The generalized Maass relations in this paper are identified to certain generalized Maass relations for Siegel-Eisenstein series of *integral weight* of degree three and of degree four. Actually this identification is a key of the proof of our result. As for generalizations of the Maass relation for Siegel-Eisenstein series of integral weight of higher degree, Yamazaki [Ya 86, Ya 89] obtained certain relations among Fourier-Jacobi coefficients of Siegel-Eisenstein series of arbitrary degree. Our generalization is different from his result, because he showed relations among Fourier-Jacobi coefficients with integer index, while our generalization in this paper concerns with the Fourier-Jacobi coefficients with 2×2 matrix index. In his paper [Ko 02] W.Kohnen gives also a generalization of the Maass relation for Siegel modular forms of even degree $2n$. However, his result is also different from our generalization, because his result is concerned with the Fourier-Jacobi coefficients with $(2n - 1) \times (2n - 1)$ matrix index.

For our purpose we generalize some results in [Ya 86, Ya 89] on Fourier-Jacobi coefficients of Siegel-Eisenstein series of integer indices to 2×2 matrix indices. Here the right-lower part of these 2×2 matrices is 1, and we need to introduce certain index-shift maps on Jacobi forms of 2×2 matrix indices. For the calculation of the action of index-shift maps on Fourier-Jacobi coefficients of Siegel-Eisenstein series, we require certain relations between Jacobi-Eisenstein series and Fourier-Jacobi coefficients of Siegel-Eisenstein series. This relation is basically shown in [Bo 83, Satz7]. We also need to show certain identity relation between Jacobi forms of integral weight of 2×2 matrix index and Jacobi forms of half-integral weight of integer index. Moreover, we need to show a compatibility between this identity relation and index-shift maps. Through these relations, the generalized Maass relation of generalized Cohen-Eisenstein series are equivalent to certain relations among Jacobi-Eisenstein series of integral weight of 2×2 matrix indices. Finally, we calculate the action of index-shift maps on Jacobi-Eisenstein series of integral weight of 2×2 matrix index. Because of this calculation, we have to

restrict ourself to Jacobi-Eisenstein series of degree one or degree two. It means we have to restrict ourself to generalized Cohen-Eisenstein series of degree two or degree three.

1.2. We explain our results more precisely. Let k be an *even* integer and $\mathcal{H}_{k-\frac{1}{2}}^{(n+1)}$ be the generalized Cohen-Eisenstein series of degree $n+1$ of weight $k-\frac{1}{2}$ (see §4.3 for the definition.)

For integer m , we denote by $e_{k,m}^{(n)}$ the m -th Fourier-Jacobi coefficient of $\mathcal{H}_{k-\frac{1}{2}}^{(n+1)}$:

$$\mathcal{H}_{k-\frac{1}{2}}^{(n+1)}\left(\begin{pmatrix} \tau & z \\ t & z \\ z & \omega \end{pmatrix}\right) = \sum_{\substack{m \geq 0 \\ m \equiv 0, 3 \pmod{4}}} e_{k,m}^{(n)}(\tau, z) e^{2\pi\sqrt{-1}m\omega},$$

where $\tau \in \mathfrak{H}_n$ and $\omega \in \mathfrak{H}_1$, and where $\mathfrak{H}_n$ denotes the Siegel upper half space of degree n . It will be shown that $e_{k,m}^{(n)}$ belongs to $J_{k-\frac{1}{2},m}^{(n)*}$ (cf. §4.3), where $J_{k-\frac{1}{2},m}^{(n)*}$ denotes a certain subspace of $J_{k-\frac{1}{2},m}^{(n)}$, and $J_{k-\frac{1}{2},m}^{(n)}$ denotes the space of all Jacobi forms of degree n of weight $k-\frac{1}{2}$ of index m (cf. §2.6). Because $\mathcal{H}_{k-\frac{1}{2}}^{(n+1)}$ belongs to the generalized plus-space (cf. §4.3), we have $e_{k,m}^{(n)} = 0$ unless $m \equiv 0, 3 \pmod{4}$. We note that we use the symbol $e_{k,m}^{(n)}$ instead of using $e_{k-\frac{1}{2},m}^{(n)}$ for the sake of simplicity. We remark that the weight of the form $e_{k,m}^{(n)}$ is not k , but $k-\frac{1}{2}$.

We define a function $e_{k,m}^{(n)}|S_p^{(n)}$ of $(\tau, z) \in \mathfrak{H}_n \times \mathbb{C}^{(n,1)}$ by

$$(e_{k,m}^{(n)}|S_p^{(n)})(\tau, z) := e_{k,mp^2}^{(n)}(\tau, z) + \left(\frac{-m}{p}\right) p^{k-2} e_{k,m}^{(n)}(\tau, pz) + p^{2k-3} e_{k,\frac{m}{p^2}}^{(n)}(\tau, p^2 z).$$

Here we regard $e_{k,\frac{m}{p^2}}^{(n)}$ as zero, if $\frac{m}{p^2}$ is not an integer or $\frac{m}{p^2} \not\equiv 0, 3 \pmod{4}$. Moreover, $\left(\frac{*}{p}\right)$ denotes the Legendre symbol for odd prime p , and $\left(\frac{a}{2}\right) := 0, 1, -1$ accordingly as a is even, $a \equiv \pm 1 \pmod{8}$ or $a \equiv \pm 3 \pmod{8}$.

For any prime p , we introduce index-shift maps $V_p^{(1)}$, $V_{1,p}^{(2)}$ and $V_{2,p}^{(2)}$ in §4.7, which are certain linear maps $V_p^{(1)} : J_{k-\frac{1}{2},m}^{(1)*} \rightarrow J_{k-\frac{1}{2},mp^2}^{(1)}$ and $V_{i,p}^{(2)} : J_{k-\frac{1}{2},m}^{(2)*} \rightarrow J_{k-\frac{1}{2},mp^2}^{(2)}$ ($i = 1, 2$). These maps are generalizations of the V_l -map in [E-Z 85, p. 43] for half-integral weight of degree 1 and of degree 2.

The problem now is to express $e_{k,m}^{(1)}|V_p^{(1)}$ and $e_{k,m}^{(2)}|V_{i,p}^{(2)}$ as linear combinations of two forms $e_{k,m}^{(n)}$ and $e_{k,m}^{(n)}|S_p^{(n)}$. For the degree $n = 1$ we obtain the following theorem.

Theorem 1.1. *For any prime p , we obtain*

$$e_{k,m}^{(1)}|V_p^{(1)} = e_{k,m}^{(1)}|S_p^{(1)}.$$

The proof of this theorem will be given in §6.

Let $\mathcal{H}_{k-\frac{1}{2}}^{(2)}(Z) = \sum_N C(N) e^{2\pi\sqrt{-1}\text{tr}(NZ)}$ be the Fourier expansion of $\mathcal{H}_{k-\frac{1}{2}}^{(2)}$, where N runs over all half-integral symmetric matrices of size 2. The identity in Theorem 1.1 is translated to relations among Fourier coefficients of $\mathcal{H}_{k-\frac{1}{2}}^{(2)}$ as follows.

Corollary 1.2. *For any prime p and for any half-integral symmetric matrix $\begin{pmatrix} n & \frac{r}{2} \\ \frac{r}{2} & m \end{pmatrix}$ of size 2, we obtain*

$$\begin{aligned} & C\left(\begin{pmatrix} np^2 & \frac{r}{2} \\ \frac{r}{2} & m \end{pmatrix}\right) + \left(\frac{-n}{p}\right) p^{k-2} C\left(\begin{pmatrix} n & \frac{r}{2p} \\ \frac{r}{2p} & m \end{pmatrix}\right) + p^{2k-3} C\left(\begin{pmatrix} \frac{n}{p^2} & \frac{r}{2p^2} \\ \frac{r}{2p^2} & m \end{pmatrix}\right) \\ = & C\left(\begin{pmatrix} n & \frac{r}{2} \\ \frac{r}{2} & mp^2 \end{pmatrix}\right) + \left(\frac{-m}{p}\right) p^{k-2} C\left(\begin{pmatrix} n & \frac{r}{2p} \\ \frac{r}{2p} & m \end{pmatrix}\right) + p^{2k-3} C\left(\begin{pmatrix} n & \frac{r}{2p^2} \\ \frac{r}{2p^2} & \frac{m}{p^2} \end{pmatrix}\right), \end{aligned}$$

where we regard $C(M)$ as 0 if the matrix M is not a half-integral symmetric matrix.

Because $e_{k,m}^{(1)}$ corresponds to a Fourier-Jacobi coefficient of Siegel-Eisenstein series of weight k of degree three, we can regard the above identity also as a certain relation among Fourier coefficients of Siegel-Eisenstein series of degree three.

As the second corollary of Theorem 1.1 we have the followings.

Corollary 1.3. *We obtain*

$$\mathcal{H}_{k-\frac{1}{2}}^{(2)}\left(\begin{pmatrix} \tau & 0 \\ 0 & \omega \end{pmatrix}\right)\Big|_{\tau} T_1(p^2) = \mathcal{H}_{k-\frac{1}{2}}^{(2)}\left(\begin{pmatrix} \tau & 0 \\ 0 & \omega \end{pmatrix}\right)\Big|_{\omega} T_1(p^2).$$

Here, in LHS we regard $\mathcal{H}_{k-\frac{1}{2}}^{(2)}\left(\begin{pmatrix} \tau & 0 \\ 0 & \omega \end{pmatrix}\right)$ as a function of $\tau \in \mathfrak{H}_1$, while we regard it as a function of $\omega \in \mathfrak{H}_1$ in RHS, and where $T_1(p^2)$ denotes the Hecke operator acting on the Kohnen plus-space (cf. [Ko 80, p.250] or §6.3).

The proof of Corollary 1.3 will be given in §6.3.

We now consider the case of degree $n = 2$. As for the Fourier-Jacobi coefficients of the generalized Cohen-Eisenstein series $\mathcal{H}_{k-\frac{1}{2}}^{(3)}$ of degree 3, we obtain the following theorem.

Theorem 1.4. *For any prime p , we obtain*

$$\begin{aligned} (e_{k,m}^{(2)}|V_{1,p}^{(2)})(\tau, z) &= p(p^{2k-5} + 1)e_{k,m}^{(2)}(\tau, pz) + (e_{k,m}^{(2)}|S_p^{(2)})(\tau, z), \\ (e_{k,m}^{(2)}|V_{2,p}^{(2)})(\tau, z) &= (p^{2k-4} - p^{2k-6})e_{k,m}^{(2)}(\tau, pz) + (p^{2k-5} + 1)(e_{k,m}^{(2)}|S_p^{(2)})(\tau, z). \end{aligned}$$

These identities can be also regarded as relations among Fourier coefficients of $\mathcal{H}_{k-\frac{1}{2}}^{(3)}$. The expression of the Fourier coefficients of $e_{k,m}^{(2)}|V_{i,p}^{(2)}$ ($i = 1, 2$) will be given in the appendix §8.2.

Because $e_{k,m}^{(2)}$ corresponds to a Fourier-Jacobi coefficient of Siegel-Eisenstein series of weight k of degree four, we can regard these identities as relations among Fourier coefficients of Siegel-Eisenstein series of degree four.

Now, let $T_{2,1}(p^2)$ and $T_{2,2}(p^2)$ be Hecke operators which generate the local Hecke ring at p acting on the generalized plus-space of degree two. These $T_{2,1}(p^2)$ and $T_{2,2}(p^2)$ are denoted as $X_1(p)$ and $p^{-k+\frac{7}{2}}X_2(p)$ in [H-I 05, p.513], respectively. As a corollary of Theorem 1.4 we have the following.

Corollary 1.5. *For any prime p we obtain*

$$\mathcal{H}_{k-\frac{1}{2}}^{(3)}\left(\begin{pmatrix} \tau & 0 \\ 0 & \omega \end{pmatrix}\right)\Big|_{\tau} T_{2,1}(p^2) = \mathcal{H}_{k-\frac{1}{2}}^{(3)}\left(\begin{pmatrix} \tau & 0 \\ 0 & \omega \end{pmatrix}\right)\Big|_{\omega} (p(p^{2k-5} + 1) + T_1(p^2)),$$

and

$$\begin{aligned} & \mathcal{H}_{k-\frac{1}{2}}^{(3)}\left(\begin{pmatrix} \tau & 0 \\ 0 & \omega \end{pmatrix}\right)\Big|_{\tau} T_{2,2}(p^2) \\ &= \mathcal{H}_{k-\frac{1}{2}}^{(3)}\left(\begin{pmatrix} \tau & 0 \\ 0 & \omega \end{pmatrix}\right)\Big|_{\omega} ((p^{2k-4} - p^{2k-6}) + p(p^{2k-5} + 1)T_1(p^2)). \end{aligned}$$

Here, in LHS of the above identities we regard $\mathcal{H}_{k-\frac{1}{2}}^{(3)}\left(\begin{pmatrix} \tau & 0 \\ 0 & \omega \end{pmatrix}\right)$ as a function of $\tau \in \mathfrak{H}_2$, while we regard it as a function of $\omega \in \mathfrak{H}_1$ in RHS. The proof of this corollary will be given in §7.3.

Remark 1.1

We remark that Tanigawa [Ta 86, §5] has obtained the same identity in Corollary 1.2 for *Siegel-Eisenstein series of half-integral weight* of degree two with arbitrary level N which satisfies $4|N$. He showed the identity by using the formula of local densities under the assumption $p \nmid N$. In our case we treat the *generalized Cohen-Eisenstein series* of degree two, which is essentially level 1. Hence, our result contains the relation also for $p = 2$.

Remark 1.2

Corollary 1.3 follows from also the pullback formula shown by Arakawa [Ar 94, Theorem 0.1], which is a certain formula with respect to Jacobi-Eisenstein series of index 1. In this paper we show Corollary 1.3 as the consequence of the generalized Maass relation of generalized Cohen-Eisenstein series of degree 3.

This paper is organized as follows: in Sect. 2, the necessary notation and definitions are introduced. In Sect. 3, the relation among Fourier-Jacobi coefficients of Siegel-Eisenstein series and Jacobi-Eisenstein series is derived. In Sect. 4, a certain map from a subspace of Jacobi forms of matrix index to a space of Jacobi forms of integer index is defined. Moreover, the compatibility of this map with certain index-shift maps is studied. In Sect. 5, we calculate the action of certain maps on the Jacobi-Eisenstein series. We express this function as a summation of certain exponential functions with generalized Gauss sums. In Sect. 6, Theorem 1.1 and Corollary 1.3 will be proved, while we will

give the proof of Theorem 1.4 and Corollary 1.5 in Sect. 7. We shall give some auxiliary calculations as an appendix in Sect. 8.

2. NOTATION AND DEFINITIONS

$\mathbb{R}^+$: the set of all positive real numbers

$R^{(n,m)}$: the set of $n \times m$ matrices with entries in a commutative ring R

Sym_n^* : the set of all half-integral symmetric matrices of size n

Sym_n^+ : all positive definite matrices in Sym_n^*

${}^t B$: the transpose of a matrix B

$A[B] := {}^t B A B$ for two matrices $A \in R^{(n,n)}$ and $B \in R^{(n,m)}$

1_n (resp. 0_n) : identity matrix (resp. zero matrix) of size n

$\text{tr}(S)$: the trace of a square matrix S

$e(S) := e^{2\pi\sqrt{-1}\text{tr}(S)}$ for a square matrix S

$\text{rank}_p(x)$: the rank of matrix $x \in \mathbb{Z}^{(n,m)}$ over the finite field $\mathbb{Z}/p\mathbb{Z}$

$\text{diag}(a_1, \dots, a_n)$: the diagonal matrix $\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$ for square matrices $a_1, \dots, a_n$

$\left(\frac{*}{p}\right)$: the Legendre symbol for odd prime p

$\left(\frac{*}{2}\right) := 0, 1, -1$ accordingly as a is even, $a \equiv \pm 1 \pmod{8}$ or $a \equiv \pm 3 \pmod{8}$

$M_{k-\frac{1}{2}}(\Gamma_0^{(n)}(4))$: the space of Siegel modular forms of weight $k - \frac{1}{2}$ of degree n

$M_{k-\frac{1}{2}}^+(\Gamma_0^{(n)}(4))$: the plus-space of $M_{k-\frac{1}{2}}(\Gamma_0^{(n)}(4))$ (cf. [Ib 92].)

$\mathfrak{H}_n$: the Siegel upper half space of degree n

$\delta(\mathcal{S}) := 1$ or 0 accordingly as the statement $\mathcal{S}$ is true or false.

2.1. Jacobi group. For a positive integer n we define the following groups:

$$\begin{aligned} \text{GSp}_n^+(\mathbb{R}) &:= \left\{ g \in \mathbb{R}^{(2n,2n)} \mid g \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix} {}^t g = n(g) \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix} \text{ for some } n(g) \in \mathbb{R}^+ \right\}, \\ \text{Sp}_n(\mathbb{R}) &:= \left\{ g \in \text{GSp}_n^+(\mathbb{R}) \mid n(g) = 1 \right\}, \\ \Gamma_n &:= \text{Sp}_n(\mathbb{R}) \cap \mathbb{Z}^{(2n,2n)}, \\ \Gamma_\infty^{(n)} &:= \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n \mid C = 0_n \right\}, \\ \Gamma_0^{(n)}(4) &:= \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n \mid C \equiv 0 \pmod{4} \right\}. \end{aligned}$$

For a matrix $g \in \text{GSp}_n^+(\mathbb{R})$, the number $n(g)$ in the above definition of $\text{GSp}_n^+(\mathbb{R})$ is called the *similitude* of the matrix g .

For positive integers n, r we define the subgroup $G_{n,r}^J \subset \mathrm{GSp}_{n+r}^+(\mathbb{R})$ by

$$G_{n,r}^J := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1_n & & \mu \\ {}^t\lambda & 1_r & {}^t\mu \\ & 1_n & -\lambda \\ & & 1_r \end{pmatrix} \in \mathrm{GSp}_{n+r}^+(\mathbb{R}) \right\},$$

where in the above definition the matrices runs over $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GSp}_n^+(\mathbb{R})$, $\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \in \mathrm{GSp}_r^+(\mathbb{R})$, $\lambda, \mu \in \mathbb{R}^{(n,r)}$ and $\kappa = {}^t\kappa \in \mathbb{R}^{(r,r)}$.

We will abbreviate such an element $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1_n & & \mu \\ {}^t\lambda & 1_r & {}^t\mu \\ & 1_n & -\lambda \\ & & 1_r \end{pmatrix}$ as

$$\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}, [(\lambda, \mu), \kappa] \right).$$

We remark that two matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$ have the same similitude in the above. We will often write

$$\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, [(\lambda, \mu), \kappa] \right)$$

instead of writing $(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times 1_{2r}, [(\lambda, \mu), \kappa])$ for simplicity. We remark that the element $(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, [(\lambda, \mu), \kappa])$ belongs to $\mathrm{Sp}_{n+r}(\mathbb{R})$. Similarly an element

$$\begin{pmatrix} 1_n & & \mu \\ {}^t\lambda & 1_r & {}^t\mu \\ & 1_n & -\lambda \\ & & 1_r \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

will be abbreviated as

$$\left([(\lambda, \mu), \kappa], \begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \right),$$

and we will abbreviate it as $([(\lambda, \mu), \kappa], \begin{pmatrix} A & B \\ C & D \end{pmatrix})$ for the case $U = V = 1_r$.

We set a subgroup $\Gamma_{n,r}^J$ of $G_{n,r}^J$ by

$$\Gamma_{n,r}^J := \left\{ (M, [(\lambda, \mu), \kappa]) \in G_{n,r}^J \mid M \in \Gamma_n, \lambda, \mu \in \mathbb{Z}^{(n,r)}, \kappa \in \mathbb{Z}^{(r,r)} \right\}.$$

2.2. The universal covering groups $\widetilde{\mathrm{GSp}}_n^+(\mathbb{R})$ and $\widetilde{G_{n,1}^J}$. We denote by $\widetilde{\mathrm{GSp}}_n^+(\mathbb{R})$ the universal covering group of $\mathrm{GSp}_n^+(\mathbb{R})$ which consists of pairs $(M, \varphi(\tau))$, where M is a matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GSp}_n^+(\mathbb{R})$, and where φ is any holomorphic function on $\mathfrak{H}_n$ such that $|\varphi(\tau)|^2 = \det(M)^{-\frac{1}{2}} |\det(C\tau + D)|$. The group operation on $\widetilde{\mathrm{GSp}}_n^+(\mathbb{R})$ is given by $(M, \varphi(\tau))(M', \varphi'(\tau)) := (MM', \varphi(M'\tau)\varphi'(\tau))$.

We embed $\Gamma_0^{(n)}(4)$ into the group $\widetilde{\mathrm{GSp}}_n^+(\mathbb{R})$ via $M \rightarrow (M, \theta^{(n)}(M\tau) \theta^{(n)}(\tau)^{-1})$, where $\theta^{(n)}(\tau) := \sum_{p \in \mathbb{Z}^{(n,1)}} e(\tau[p])$ is the theta constant. We denote by $\Gamma_0^{(n)}(4)^*$ the image of $\Gamma_0^{(n)}(4)$ in $\widetilde{\mathrm{GSp}}_n^+(\mathbb{R})$ by this embedding.

We define the Heisenberg group

$$H_{n,1}(\mathbb{R}) := \{ (1_{2n}, [(\lambda, \mu), \kappa]) \in \mathrm{Sp}_{n+1}(\mathbb{R}) \mid \lambda, \mu \in \mathbb{R}^{(n,1)}, \kappa \in \mathbb{R} \}.$$

If there is no confusion, we will write $[(\lambda, \mu), \kappa]$ for the element $(1_{2n}, [(\lambda, \mu), \kappa])$ for simplicity.

We define the group

$$\begin{aligned} \widetilde{G}_{n,1}^J &:= \widetilde{\mathrm{GSp}}_n^+(\mathbb{R}) \ltimes H_{n,1}(\mathbb{R}) \\ &= \left\{ (\tilde{M}, [(\lambda, \mu), \kappa]) \mid \tilde{M} \in \widetilde{\mathrm{GSp}}_n^+(\mathbb{R}), [(\lambda, \mu), \kappa] \in H_{n,1}(\mathbb{R}) \right\}. \end{aligned}$$

Here the group operation on $\widetilde{G}_{n,1}^J$ is given by

$$(\tilde{M}_1, [(\lambda_1, \mu_1), \kappa_1]) \cdot (\tilde{M}_2, [(\lambda_2, \mu_2), \kappa_2]) := (\tilde{M}_1 \tilde{M}_2, [(\lambda', \mu'), \kappa'])$$

for $(\tilde{M}_i, [(\lambda_i, \mu_i), \kappa_i]) \in \widetilde{G}_{n,1}^J$ ($i = 1, 2$), and where $[(\lambda', \mu'), \kappa'] \in H_{n,1}(\mathbb{R})$ is the matrix determined through the identity

$$\begin{aligned} &(M_1 \times \begin{pmatrix} n(M_1) & 0 \\ 0 & 1 \end{pmatrix}, [(\lambda_1, \mu_1), \kappa_1]) (M_2 \times \begin{pmatrix} n(M_2) & 0 \\ 0 & 1 \end{pmatrix}, [(\lambda_2, \mu_2), \kappa_2]) \\ &= (M_1 M_2 \times \begin{pmatrix} n(M_1)n(M_2) & 0 \\ 0 & 1 \end{pmatrix}, [(\lambda', \mu'), \kappa']) \end{aligned}$$

in $G_{n,1}^J$. Here $n(M_i)$ is the similitude of M_i .

2.3. Action of the Jacobi group. The group $G_{n,r}^J$ acts on $\mathfrak{H}_n \times \mathbb{C}^{(n,r)}$ by

$$\gamma \cdot (\tau, z) := \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \tau, {}^t(C\tau + D)^{-1}(z + \tau\lambda + \mu){}^tU \right)$$

for any $\gamma = ((\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}) \times (\begin{smallmatrix} U & 0 \\ 0 & V \end{smallmatrix}), [(\lambda, \mu), \kappa]) \in G_{n,r}^J$ and for any $(\tau, z) \in \mathfrak{H}_n \times \mathbb{C}^{(n,r)}$. Here $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \tau := (A\tau + B)(C\tau + D)^{-1}$ is the usual transformation.

The group $\widetilde{G}_{n,1}^J$ acts on $\mathfrak{H}_n \times \mathbb{C}^{(n,1)}$ through the action of $G_{n,1}^J$ by

$$\tilde{\gamma} \cdot (\tau, z) := (M \times \begin{pmatrix} n(M) & 0 \\ 0 & 1 \end{pmatrix}, [(\lambda, \mu), \kappa]) \cdot (\tau, z)$$

for $\tilde{\gamma} = ((M, \varphi), [(\lambda, \mu), \kappa]) \in \widetilde{G}_{n,1}^J$ and for $(\tau, z) \in \mathfrak{H}_n \times \mathbb{C}^{(n,1)}$. Here $n(M)$ is the similitude of $M \in \mathrm{GSp}_n^+(\mathbb{R})$.

2.4. Factors of automorphy. Let k be an integer and let $\mathcal{M} \in \text{Sym}_r^+$. For $\gamma = \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}, [(\lambda, \mu), \kappa] \right) \in G_{n,r}^J$ we define a factor of automorphy

$$J_{k,\mathcal{M}}(\gamma, (\tau, z)) := \det(V)^k \det(C\tau + D)^k e(V^{-1}\mathcal{M}U(((C\tau + D)^{-1}C)[z + \tau\lambda + \mu])) \\ \times e(-V^{-1}\mathcal{M}U({}^t\lambda\tau\lambda + {}^t z\lambda + {}^t\lambda z + {}^t\mu\lambda + {}^t\lambda\mu + \kappa)).$$

We define a slash operator $|_{k,\mathcal{M}}$ by

$$(\phi|_{k,\mathcal{M}}\gamma)(\tau, z) := J_{k,\mathcal{M}}(\gamma, (\tau, z))^{-1} \phi(\gamma \cdot (\tau, z))$$

for any function ϕ on $\mathfrak{H}_n \times \mathbb{C}^{(n,r)}$ and for any $\gamma \in G_{n,r}^J$. We remark that

$$J_{k,\mathcal{M}}(\gamma_1\gamma_2, (\tau, z)) = J_{k,\mathcal{M}}(\gamma_1, \gamma_2 \cdot (\tau, z)) J_{k,V_1^{-1}\mathcal{M}U_1}(\gamma_2, (\tau, z)), \\ \phi|_{k,\mathcal{M}}\gamma_1\gamma_2 = (\phi|_{k,\mathcal{M}}\gamma_1)|_{k,V_1^{-1}\mathcal{M}U_1}\gamma_2.$$

for any $\gamma_i = \left(M_i \times \begin{pmatrix} U_i & 0 \\ 0 & V_i \end{pmatrix}, [(\lambda_i, \mu_i), \kappa_i] \right) \in G_{n,r}^J$ ($i = 1, 2$).

Let k and m be integers. We define a slash operator $|_{k-\frac{1}{2},m}$ for any function ϕ on $\mathfrak{H}_n \times \mathbb{C}^{(n,1)}$ by

$$\phi|_{k-\frac{1}{2},m}\tilde{\gamma} := J_{k-\frac{1}{2},m}(\tilde{\gamma}, (\tau, z))^{-1} \phi(\tilde{\gamma} \cdot (\tau, z))$$

for any $\tilde{\gamma} = ((M, \varphi), [(\lambda, \mu), \kappa]) \in \widetilde{G_{n,1}^J}$. Here we define a factor of automorphy

$$J_{k-\frac{1}{2},m}(\tilde{\gamma}, (\tau, z)) := \varphi(\tau)^{2k-1} e(n(M)m(((C\tau + D)^{-1}C)[z + \tau\lambda + \mu])) \\ \times e(-n(M)m({}^t\lambda\tau\lambda + {}^t z\lambda + {}^t\lambda z + {}^t\mu\lambda + {}^t\lambda\mu + \kappa)),$$

where $n(M)$ is the similitude of M . We remark that

$$J_{k-\frac{1}{2},m}(\tilde{\gamma}_1\tilde{\gamma}_2, (\tau, z)) = J_{k-\frac{1}{2},m}(\tilde{\gamma}_1, \tilde{\gamma}_2 \cdot (\tau, z)) J_{k-\frac{1}{2},n(M_1)m}(\tilde{\gamma}_2, (\tau, z)) \\ \phi|_{k-\frac{1}{2},m}\tilde{\gamma}_1\tilde{\gamma}_2 = (\phi|_{k-\frac{1}{2},m}\tilde{\gamma}_1)|_{k-\frac{1}{2},n(M_1)m}\tilde{\gamma}_2$$

for any $\tilde{\gamma}_i = ((M_i, \varphi_i), [(\lambda_i, \mu_i), \kappa_i]) \in \widetilde{G_{n,1}^J}$ ($i = 1, 2$).

2.5. Jacobi forms of matrix index. We quote the definition of Jacobi forms of matrix index from [Zi 89]. For an integer k and for an matrix $\mathcal{M} \in \text{Sym}_r^+$ a $\mathbb{C}$ -valued holomorphic function ϕ on $\mathfrak{H}_n \times \mathbb{C}^{(n,r)}$ is called a *Jacobi form of weight k of index $\mathcal{M}$ of degree n* , if ϕ satisfies the following two conditions:

- (1) the transformation formula $\phi|_{k,\mathcal{M}}\gamma = \phi$ for any $\gamma \in \Gamma_{n,r}^J$,
- (2) ϕ has the Fourier expansion: $\phi(\tau, z) = \sum_{\substack{N \in \text{Sym}_n^*, R \in Z^{(n,r)} \\ 4N - R\mathcal{M}^{-1}R \geq 0}} c(N, R) e(N\tau) e({}^t Rz).$

We remark that the second condition follows from the Koecher principle (cf. [Zi 89, Lemma 1.6]) if $n > 1$.

We denote by $J_{k,\mathcal{M}}^{(n)}$ the $\mathbb{C}$ -vector space of Jacobi forms of weight k of index $\mathcal{M}$ of degree n .

2.6. Jacobi forms of half-integral weight. We set the subgroup $\Gamma_{n,1}^{J*}$ of $\widetilde{G_{n,1}^J}$ by

$$\begin{aligned}\Gamma_{n,1}^{J*} &:= \left\{ (M^*, [(\lambda, \mu), \kappa]) \in \widetilde{G_{n,1}^J} \mid M^* \in \Gamma_0^{(n)}(4)^*, \lambda, \mu \in \mathbb{Z}^{(n,1)}, \kappa \in \mathbb{Z} \right\} \\ &\cong \Gamma_0^{(n)}(4)^* \ltimes H_{n,1}(\mathbb{Z}),\end{aligned}$$

where we put $H_{n,1}(\mathbb{Z}) := H_{n,1}(\mathbb{R}) \cap \mathbb{Z}^{(2n+2, 2n+2)}$. Here the group $\Gamma_0^{(n)}(4)^*$ was defined in §2.2.

For an integer k and for an integer m , a holomorphic function ϕ on $\mathfrak{H}_n \times \mathbb{C}^{(n,1)}$ is called a *Jacobi form of weight $k - \frac{1}{2}$ of index m* , if ϕ satisfies the following two conditions:

- (1) the transformation formula $\phi|_{k-\frac{1}{2}, m} \gamma^* = \phi$ for any $\gamma^* \in \Gamma_{n,1}^{J*}$,
- (2) $\phi^2|_{2k-1, 2m} \gamma$ has the Fourier expansion for any $\gamma \in \Gamma_{n,1}^J$:

$$(\phi^2|_{2k-1, 2m} \gamma)(\tau, z) = \sum_{\substack{N \in \text{Sym}_n^*, R \in \mathbb{Z}^{(n,1)} \\ 4Nm - R^t R \geq 0}} C(N, R) e\left(\frac{1}{h} N \tau\right) e\left(\frac{1}{h} {}^t R z\right)$$

with a certain integer $h > 0$, and where the slash operator $|_{2k-1, 2m}$ was defined in §2.5.

We denote by $J_{k-\frac{1}{2}, m}^{(n)}$ the $\mathbb{C}$ -vector space of Jacobi forms of weight $k - \frac{1}{2}$ of index m .

2.7. Index-shift maps. In this subsection we introduce two kinds of maps. The both maps shift the index of Jacobi forms and these are generalizations of the V_l -map in the book of Eichler-Zagier [E-Z 85].

We define two groups $\text{GSp}_n^+(\mathbb{Z}) := \text{GSp}_n^+(\mathbb{R}) \cap \mathbb{Z}^{(2n, 2n)}$ and

$$\widetilde{\text{GSp}_n^+(\mathbb{Z})} := \left\{ (M, \varphi) \in \widetilde{\text{GSp}_n^+(\mathbb{R})} \mid M \in \text{GSp}_n^+(\mathbb{Z}) \right\}.$$

First we define index-shift maps for Jacobi forms of *integral weight of matrix index*. Let $\mathcal{M} = \begin{pmatrix} * & * \\ * & 1 \end{pmatrix} \in \text{Sym}_2^+$. Let $X \in \text{GSp}_n^+(\mathbb{Z})$ be a matrix such that the similitude of X is $n(X) = p^2$ with a prime p . For any $\phi \in J_{k, \mathcal{M}}^{(n)}$ we define the map

$$\phi|V(X) := \sum_{u, v \in (\mathbb{Z}/p\mathbb{Z})^{(n,1)}} \sum_{M \in \Gamma_n \backslash \Gamma_n X \Gamma_n} \phi|_{k, \mathcal{M}} \left(M \times \begin{pmatrix} p^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, [((0, u), (0, v)), 0_n] \right),$$

where $(0, u), (0, v) \in (\mathbb{Z}/p\mathbb{Z})^{(n,2)}$. The above summation is a finite sum and do not depend on the choice of the representatives u, v and M . A straightforward calculation shows that $\phi|V(X)$ belongs to $J_{k, \mathcal{M}}^{(n)} \left[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right]$. Namely $V(X)$ is a map:

$$V(X) : J_{k, \mathcal{M}}^{(n)} \rightarrow J_{k, \mathcal{M}}^{(n)} \left[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right].$$

For the sake of simplicity we set

$$V_{\alpha, n-\alpha}(p^2) := V(\text{diag}(1_\alpha, p1_{n-\alpha}, p^2 1_\alpha, p1_{n-\alpha}))$$

for any prime p and for any α ($0 \leq \alpha \leq n$).

Next we shall define index-shift maps for Jacobi forms of *half-integral weight of integer index*. We assume that p is an odd prime. Let m be a positive integer. Let $Y = (X, \varphi) \in \widetilde{\text{GSp}}_n^+(\mathbb{Z})$ be a matrix such that the similitude of X is $n(X) = p^2$. For $\psi \in J_{k-\frac{1}{2}, m}^{(n)}$ we define the map

$$\psi| \widetilde{V}(Y) := n(X)^{\frac{n(2k-1)}{4} - \frac{n(n+1)}{2}} \sum_{\tilde{M} \in \Gamma_0^{(n)}(4)^* \backslash \Gamma_0^{(n)}(4)^* Y \Gamma_0^{(n)}(4)^*} \psi|_{k-\frac{1}{2}, m}(\tilde{M}, [(0, 0), 0_n]),$$

where the above summation is a finite sum and does not depend on the choice of the representatives $\tilde{M}$. A direct computation shows that $\psi| \widetilde{V}(Y)$ belongs to $J_{k-\frac{1}{2}, mp^2}^{(n)}$. For the sake of simplicity we set

$$\tilde{V}_{\alpha, n-\alpha}(p^2) := \tilde{V}((\text{diag}(1_\alpha, p1_{n-\alpha}, p^2 1_\alpha, p1_{n-\alpha}), p^{\alpha/2}))$$

for any odd prime p and for any α ($0 \leq \alpha \leq n$).

As for $p = 2$, we will introduce index-shift maps $\tilde{V}_{\alpha, n-\alpha}(4)$ in §4.6, which are maps from a certain subspace $J_{k-\frac{1}{2}, m}^{(n)*}$ of $J_{k-\frac{1}{2}, m}^{(n)}$ to $J_{k-\frac{1}{2}, 4m}^{(n)}$.

3. FOURIER JACOBI EXPANSION OF SIEGEL-EISENSTEIN SERIES WITH MATRIX INDEX

In this section we assume that k is an even integer.

For $\mathcal{M} \in \text{Sym}_2^+$ and for an even integer k we define the Jacobi-Eisenstein series of weight k of index $\mathcal{M}$ by

$$E_{k, \mathcal{M}}^{(n)} := \sum_{M \in \Gamma_\infty^{(n)} \backslash \Gamma_n} \sum_{\lambda \in \mathbb{Z}^{(n, 2)}} 1|_{k, \mathcal{M}}([\lambda, 0], M).$$

The Jacobi-Eisenstein series $E_{k, \mathcal{M}}^{(n)}$ is absolutely convergent for $k > n + 3$ (cf. [Zi 89]).

The Siegel-Eisenstein series $E_k^{(n)}$ of weight k of degree n is defined by

$$E_k^{(n)}(Z) := \sum_{(C, D)} \det(CZ + D)^{-k},$$

where $Z \in \mathfrak{H}_n$ and (C, D) runs over a complete set of representatives of the equivalence classes of coprime symmetric pairs of size n . Let

$$E_k^{(n)}\left(\begin{pmatrix} \tau & z \\ t & \omega \end{pmatrix}\right) = \sum_{\mathcal{M} \in \text{Sym}_2^*} e_{k, \mathcal{M}}^{(n-2)}(\tau, z) e(\mathcal{M}\omega)$$

be the Fourier-Jacobi expansion of the Siegel-Eisenstein series $E_k^{(n)}$ of weight k of degree n , where $\tau \in \mathfrak{H}_{n-2}$, $\omega \in \mathfrak{H}_2$ and $z \in \mathbb{C}^{(n-2,2)}$.

The explicit formula for the Fourier-Jacobi expansion of Siegel-Eisenstein series is given in [Bo 83, Satz 7] for arbitrary degree. The purpose of this section is to express the Fourier-Jacobi coefficient $e_{k,\mathcal{M}}^{(n-2)}$ for $\mathcal{M} = \begin{pmatrix} * & * \\ * & 1 \end{pmatrix} \in \text{Sym}_2^+$ as a summation of Jacobi-Eisenstein series of matrix index (Proposition 3.3.)

First, we obtain the following lemma.

Lemma 3.1. *For any $A \in GL_n(\mathbb{Z})$ we have*

$$E_{k,\mathcal{M}}^{(n)}(\tau, z) = E_{k,\mathcal{M}[A^{-1}]}^{(n)}(\tau, z^t A)$$

and

$$e_{k,\mathcal{M}}^{(n)}(\tau, z) = e_{k,\mathcal{M}[A^{-1}]}^{(n)}(\tau, z^t A).$$

Proof. The first identity follows directly from the definition. The transformation formula $E_k^{(n+2)} \left(\begin{pmatrix} 1_n & \\ & A \end{pmatrix} \begin{pmatrix} \tau & z \\ t z & \omega \end{pmatrix} \begin{pmatrix} 1_n & \\ & t A \end{pmatrix} \right) = E_k^{(n+2)} \left(\begin{pmatrix} \tau & z \\ t z & \omega \end{pmatrix} \right)$ gives the second identity. $\square$

Let m be a positive integer. We denote by D_0 the discriminant of $\mathbb{Q}(\sqrt{-m})$, and we put $f := \sqrt{\frac{m}{|D_0|}}$. We note that f is a positive integer if $-m \equiv 0, 1 \pmod{4}$.

We denote by $h_{k-\frac{1}{2}}(m)$ the m -th Fourier coefficient of the Cohen-Eisenstein series of weight $k - \frac{1}{2}$ (cf. Cohen [Co 75]). The following formula is known (cf. [Co 75], [E-Z 85]):

$$\begin{aligned} & h_{k-\frac{1}{2}}(m) \\ &= \begin{cases} h_{k-\frac{1}{2}}(|D_0|) m^{k-\frac{3}{2}} \sum_{d|f} \mu(d) \left(\frac{D_0}{d}\right) d^{1-k} \sigma_{3-2k}\left(\frac{f}{d}\right), & \text{if } -m \equiv 0, 1 \pmod{4}, \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

where we define $\sigma_a(b) := \sum_{d|b} d^a$.

We assume $-m \equiv 0, 1 \pmod{4}$. Let D_0 and f be as above. We define

$$g_k(m) := \sum_{d|f} \mu(d) h_{k-\frac{1}{2}}\left(\frac{m}{d^2}\right).$$

We will use the following lemma for the proof of Theorem 1.1 and Theorem 1.4.

Lemma 3.2. *Let m be a natural number such that $-m \equiv 0, 1 \pmod{4}$. Then for any prime p we have*

$$g_k(p^2 m) = \left(p^{2k-3} - \left(\frac{-m}{p} \right) p^{k-2} \right) g_k(m).$$

Proof. Let D_0, f be as above. We have

$$h_{k-\frac{1}{2}}(m) = h_{k-\frac{1}{2}}(|D_0|)|D_0|^{k-\frac{3}{2}} \prod_{q|f} \left\{ \sigma_{2k-3}(q^{l_q}) - \left(\frac{D_0}{q} \right) q^{k-2} \sigma_{2k-3}(q^{l_q-1}) \right\},$$

where q runs over all primes which divide f , and where we put $l_q := \text{ord}_q(f)$. In particular, the function $h_{k-\frac{1}{2}}(m)(h_{k-\frac{1}{2}}(|D_0|)|D_0|^{k-\frac{3}{2}})^{-1}$ is multiplicative with respect to f . We have also

$$\begin{aligned} & h_{k-\frac{1}{2}}(|D_0|q^{2l_q}) - h_{k-\frac{1}{2}}(|D_0|q^{2l_q-2}) \\ &= h_{k-\frac{1}{2}}(|D_0|)|D_0|^{k-\frac{3}{2}} \left(q^{(2k-3)l_q} - \left(\frac{D_0}{q} \right) q^{k-2+(2k-3)(l_q-1)} \right), \end{aligned}$$

Thus

$$\begin{aligned} g_k(m) &= h_{k-\frac{1}{2}}(|D_0|)|D_0|^{k-\frac{3}{2}} \prod_{q|f} \frac{h_{k-\frac{1}{2}}(|D_0|q^{2l_q}) - h_{k-\frac{1}{2}}(|D_0|q^{2l_q-2})}{h_{k-\frac{1}{2}}(|D_0|)|D_0|^{k-\frac{3}{2}}} \\ &= h_{k-\frac{1}{2}}(|D_0|)|D_0|^{k-\frac{3}{2}} \prod_{q|f} \left(q^{(2k-3)l_q} - \left(\frac{D_0}{q} \right) q^{k-2+(2k-3)(l_q-1)} \right). \end{aligned}$$

The lemma follows from this identity, because $\left(\frac{-m}{p} \right) = 0$ if $p|f$; $\left(\frac{-m}{p} \right) = \left(\frac{D_0}{p} \right)$ if $p \nmid f$. $\square$

We obtain the following proposition.

Proposition 3.3. *For $\mathcal{M} = \begin{pmatrix} * & * \\ * & 1 \end{pmatrix} \in \text{Sym}_2^+$ we put $m = \det(2\mathcal{M})$. Let D_0, f be as above. If $k > n + 1$, then*

$$e_{k,\mathcal{M}}^{(n-2)}(\tau, z) = \sum_{d|f} g_k\left(\frac{m}{d^2}\right) E_{k,\mathcal{M}[{}^t W_d^{-1}]}^{(n-2)}(\tau, z W_d),$$

where we chose a matrix $W_d \in GL_2(\mathbb{Q}) \cap \mathbb{Z}^{(2,2)}$ for each d which satisfies the conditions $\det(W_d) = d$ and $W_d^{-1} \mathcal{M} {}^t W_d^{-1} = \begin{pmatrix} * & * \\ * & 1 \end{pmatrix} \in \text{Sym}_2^+$. The above summation is independent of the choice of the matrix W_d .

Proof. The Satz 7 in [Bo 83] is the essential part of this proof. For $\mathcal{M}' \in \text{Sym}_n^+$ we denote by $a_2^k(\mathcal{M}')$ the $\mathcal{M}'$ -th Fourier coefficient of Siegel-Eisenstein series of weight k of degree 2. We put

$$\text{M}_2^n(\mathbb{Z})^* := \left\{ N \in \mathbb{Z}^{(2,2)} \mid \det(N) \neq 0 \text{ and there exists } V = \begin{pmatrix} N & * \\ * & * \end{pmatrix} \in GL_n(\mathbb{Z}) \right\}.$$

We call a matrix $N \in \mathbb{Z}^{(n,2)}$ *primitive* if there exists a matrix $V \in \mathrm{GL}_n(\mathbb{Z})$ such that $V = (N \ *)$. From [Bo 83, Satz 7] we have

$$e_{k,\mathcal{M}}^{(n-2)}(\tau, z) = \sum_{\substack{N_1 \in M_2^n(\mathbb{Z})^* / \mathrm{GL}(2, \mathbb{Z}) \\ N_1^{-1} \mathcal{M}^t N_1^{-1} \in \mathrm{Sym}_2^+}} a_2^k(\mathcal{M} [{}^t N_1^{-1}]) \sum_{\substack{N_3 \in \mathbb{Z}^{(n-2,2)} \\ \begin{pmatrix} N_1 \\ N_3 \end{pmatrix} : \text{primitive}}} f(\mathcal{M}, N_1, N_3; \tau, z),$$

where we define

$$\begin{aligned} & f(\mathcal{M}, N_1, N_3; \tau, z) \\ := & \sum_{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_\infty^{(n-2)} \setminus \Gamma_{n-2}} \det(C\tau + D)^{-k} \\ & \times e(\mathcal{M} \{ -{}^t z(C\tau + D)^{-1} C z + {}^t z(C\tau + D)^{-1} N_3 N_1^{-1} \\ & + {}^t N_1^{-1} N_3 {}^t (C\tau + D)^{-1} z + {}^t N_1^{-1} N_3 (A\tau + B)(C\tau + D)^{-1} N_3 N_1^{-1} \}). \end{aligned}$$

For positive integer l we chose a matrix $W_l \in \mathbb{Z}^{(2,2)}$ which satisfies three conditions $\det(W_l) = l$, $W_l^{-1} \mathcal{M}^t W_l^{-1} \in \mathrm{Sym}_2^+$ and $W_l^{-1} \mathcal{M}^t W_l^{-1} = \begin{pmatrix} * & * \\ * & 1 \end{pmatrix}$. Because of these conditions, W_l has the form $W_l = \begin{pmatrix} l & x \\ 0 & 1 \end{pmatrix}$ with some $x \in \mathbb{Z}$. The set $W_l \mathrm{GL}(2, \mathbb{Z})$ is uniquely determined for each positive integer l such that $l^2 | m$.

Thus

$$\begin{aligned} & e_{k,\mathcal{M}}^{(n-2)}(\tau, z) \\ = & \sum_{\substack{l \\ l^2 | m}} a_2^k(\mathcal{M} [{}^t W_l^{-1}]) \sum_{a|l} \mu(a) \sum_{N_3 \in \mathbb{Z}^{(n-2,2)}} f(\mathcal{M}, W_l, N_3 \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}; \tau, z) \\ = & \sum_{\substack{l \\ l^2 | m}} a_2^k(\mathcal{M} [{}^t W_l^{-1}]) \sum_{a|l} \mu(a) \sum_{N_3 \in \mathbb{Z}^{(n-2,2)}} f(\mathcal{M} [{}^t W_l^{-1} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}], 1_2, N_3; \tau, z W_l \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}^{-1}) \end{aligned}$$

Therefore

$$\begin{aligned} & e_{k,\mathcal{M}}^{(n-2)}(\tau, z) \\ = & \sum_{\substack{l \\ l^2 | m}} a_2^k(\mathcal{M} [{}^t W_l^{-1}]) \sum_{a|l} \mu(a) E_{k,\mathcal{M} [{}^t W_l^{-1} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}]}^{(n-2)}(\tau, z W_l \begin{pmatrix} a^{-1} & \\ & 1 \end{pmatrix}) \\ = & \sum_{\substack{d \\ d^2 | m}} E_{k,\mathcal{M} [{}^t W_d^{-1}]}^{(n-2)}(\tau, z W_d) \sum_{\substack{a \\ a^2 | \frac{m}{d^2}}} \mu(a) a_2^k(\mathcal{M} [{}^t W_d^{-1} \begin{pmatrix} a^{-1} & \\ & 1 \end{pmatrix}])). \end{aligned}$$

Here we have $a_2^k(\mathcal{M}') = h_{k-\frac{1}{2}}(\det(2\mathcal{M}'))$ for any $\mathcal{M}' = \begin{pmatrix} * & * \\ * & 1 \end{pmatrix} \in \text{Sym}_2^+$. Moreover, if $m \not\equiv 0, 3 \pmod{4}$, then $h_{k-\frac{1}{2}}(m) = 0$. Hence

$$e_{k,\mathcal{M}}^{(n-2)}(\tau, z) = \sum_{\substack{d \\ d|f}} E_{k,\mathcal{M}[\iota W_d^{-1}]}^{(n-2)}(\tau, z W_d) \sum_{\substack{a \\ a|\frac{f}{d}}} \mu(a) h_{k-\frac{1}{2}}\left(\frac{m}{a^2 d^2}\right).$$

Therefore the proposition follows. $\square$

4. RELATION BETWEEN JACOBI FORMS OF INTEGER INDEX AND OF MATRIX INDEX

In this section we fix a positive definite half-integral symmetric matrix $\mathcal{M} \in \text{Sym}_2^+$, and we assume that $\mathcal{M}$ has a form $\mathcal{M} = \begin{pmatrix} l & \frac{1}{2}r \\ \frac{1}{2}r & 1 \end{pmatrix}$ with integers l and r .

The purpose of this section is to give a map $\iota_{\mathcal{M}}$, which is a map from certain holomorphic functions on $\mathfrak{H}_n \times \mathbb{C}^{(n,2)}$ to holomorphic functions on $\mathfrak{H}_n \times \mathbb{C}^{(n,1)}$.

The restriction of $\iota_{\mathcal{M}}$ gives a map from a certain subspaces $J_{k,\mathcal{M}}^{(n)*}$ of $J_{k,\mathcal{M}}^{(n)}$ to a certain subspace $J_{k-\frac{1}{2},\det(2\mathcal{M})}^{(n)*}$ of $J_{k-\frac{1}{2},\det(2\mathcal{M})}^{(n)}$ (cf. Lemma 4.2). Moreover, we shall show the compatibility of the restriction of this map $\iota_{\mathcal{M}}$ with index-shift maps which shift indices of spaces of Jacobi forms. (cf. Proposition 4.3 and Proposition 4.4). Furthermore we define index-shift maps for $J_{k-\frac{1}{2},\det(2\mathcal{M})}^{(n)*}$ at $p = 2$ through the map $\iota_{\mathcal{M}}$ (cf. §4.6).

4.1. An expansion of Jacobi forms of integer index. In this subsection we shall introduce certain spaces $M_k^*(\Gamma_n)$, $J_{k,1}^{(n-1)*}$ and $J_{k,\mathcal{M}}^{(n-2)*}$. Moreover, we consider an expansion of Jacobi forms of integer index.

The $\mathbb{C}$ -vector subspace $M_k^*(\Gamma_n)$ of $M_k(\Gamma_n)$ denotes the image of the Ikeda lifts in $M_k(\Gamma_n)$. We remark that this subspace $M_k^*(\Gamma_n)$ contains the Siegel-Eisenstein series $E_k^{(n)}$. In the case $n = 2$, the space $M_k^*(\Gamma_2)$ coincides the Maass space.

We denote by $J_{k,1}^{(n)}$ the space of Jacobi forms of weight k of index 1 of degree n (cf. §2.5). The $\mathbb{C}$ -vector subspace $J_{k,1}^{(n-1)*}$ of $J_{k,1}^{(n-1)}$ denotes the image of $M_k^*(\Gamma_n)$ through the Fourier-Jacobi expansion with index 1. Moreover, the $\mathbb{C}$ -vector subspace $J_{k,\mathcal{M}}^{(n-2)*}$ of $J_{k,\mathcal{M}}^{(n-2)}$ denotes the image of $M_k^*(\Gamma_n)$ in $J_{k,\mathcal{M}}^{(n-2)}$ through the Fourier-Jacobi expansion with index $\mathcal{M}$, where $\mathcal{M}$ is a 2×2 matrix.

Let $\phi_1(\tau, z) \in J_{k,1}^{(n-1)}$ be a Jacobi form of index 1. We regard $\phi_1(\tau, z) e(\omega)$ as a holomorphic function on $\mathfrak{H}_n$, where $\tau \in \mathfrak{H}_{n-1}$, $z \in \mathbb{C}^{(n-1,1)}$ and $\omega \in \mathfrak{H}_1$ such that $\begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in \mathfrak{H}_n$. We have an expansion

$$\phi_1(\tau, z) e(\omega) = \sum_{\substack{S \in \text{Sym}_2^+ \\ S = \begin{pmatrix} * & * \\ * & 1 \end{pmatrix}}} \phi_S(\tau', z') e(S\omega'),$$

where $\tau' \in \mathfrak{H}_{n-2}$, $z' \in \mathbb{C}^{(n-2,2)}$ and $\omega' \in \mathfrak{H}_2$ such that $(\begin{smallmatrix} \tau & z \\ \omega \end{smallmatrix}) = (\begin{smallmatrix} \tau' & z' \\ \omega' \end{smallmatrix}) \in \mathfrak{H}_n$. Because the group $\Gamma_{n-2,2}^J$ is a subgroup of $\Gamma_{n-1,1}^J$, the form ϕ_S belongs to $J_{k,S}^{(n-2)}$. We denote this map by $\text{FJ}_{1,S}$, namely we have a map

$$\text{FJ}_{1,S} : J_{k,1}^{(n-1)} \rightarrow J_{k,S}^{(n-2)}.$$

4.2. Fourier-Jacobi expansion of Siegel modular forms of half-integral weight.
The purpose of this subsection is to show the following lemma.

Lemma 4.1. *Let $F(\begin{smallmatrix} \tau & z \\ \omega \end{smallmatrix}) = \sum_{m \in \mathbb{Z}} \phi_m(\tau, z) e(m\omega)$ be a Fourier-Jacobi expansion of $F \in M_{k-\frac{1}{2}}(\Gamma_0^{(n+1)}(4))$, where $\tau \in \mathfrak{H}_n$, $\omega \in \mathfrak{H}_1$ and $z \in \mathbb{C}^{(n,1)}$. Then $\phi_m \in J_{k-\frac{1}{2},m}^{(n)}$ for any natural number m .*

Proof. Due to the definition of $J_{k-\frac{1}{2},m}^{(n)}$, it is enough to show the identity

$$\theta^{(n+1)}(\gamma \cdot (\begin{smallmatrix} \tau & z \\ \omega \end{smallmatrix})) \theta^{(n+1)}((\begin{smallmatrix} \tau & z \\ \omega \end{smallmatrix}))^{-1} = \theta^{(n)}((\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}) \cdot \tau) \theta^{(n)}(\tau)^{-1}$$

for any $\gamma = ((\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}), [(\lambda, \mu), \kappa]) \in \Gamma_{n,1}^J$, $(\begin{smallmatrix} \tau & z \\ \omega \end{smallmatrix}) \in \mathfrak{H}_{n+1}$ such that $\tau \in \mathfrak{H}_n$, $\omega \in \mathfrak{H}_1$. Here $\theta^{(n+1)}$ and $\theta^{(n)}$ are the theta constants (cf. §2.2.)

For any $M = (\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}) \in \Gamma_0^{(n+1)}(4)$, it is known that

$$(\theta^{(n+1)}(M \cdot Z) \theta^{(n+1)}(Z)^{-1})^2 = \det(C'Z + D') \left(\frac{-4}{\det D'} \right),$$

where $Z \in \mathfrak{H}_{n+1}$. Here $(\frac{-4}{\det D'})$ ($= (-1)^{\frac{\det D' - 1}{2}}$) is the quadratic symbol. Hence for any $\gamma = ((\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}), [(\lambda, \mu), \kappa]) \in \Gamma_{n,1}^J$ we obtain

$$(\theta^{(n+1)}(\gamma \cdot Z) \theta^{(n+1)}(Z)^{-1})^2 = \det(C\tau + D) \left(\frac{-4}{\det D} \right),$$

where $Z = (\begin{smallmatrix} \tau & z \\ \omega \end{smallmatrix}) \in \mathfrak{H}_{n+1}$ with $\tau \in \mathfrak{H}_n$. In particular, the holomorphic function $\frac{\theta^{(n+1)}(\gamma \cdot Z)}{\theta^{(n+1)}(Z)}$ does not depend on the choice of $z \in \mathbb{C}^{(n,1)}$ and of $\omega \in \mathfrak{H}_1$. We substitute $z = 0$ into $\frac{\theta^{(n+1)}(\gamma \cdot Z)}{\theta^{(n+1)}(Z)}$ and a straightforward calculation shows

$$\frac{\theta^{(n+1)}(\gamma \cdot (\begin{smallmatrix} \tau & 0 \\ \omega \end{smallmatrix}))}{\theta^{(n+1)}((\begin{smallmatrix} \tau & 0 \\ \omega \end{smallmatrix}))} = \frac{\theta^{(n)}((\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}) \cdot \tau)}{\theta^{(n)}(\tau)}.$$

Hence we conclude this lemma. $\square$

4.3. The map σ and the subspace $J_{k-\frac{1}{2},m}^{(n)*}$. In this subsection we introduce generalized Cohen-Eisenstein series $\mathcal{H}_{k-\frac{1}{2}}^{(n+1)}$ and consider the Fourier-Jacobi expansion of $\mathcal{H}_{k-\frac{1}{2}}^{(n+1)}$. Moreover, we will introduce a subspace $J_{k-\frac{1}{2},m}^{(n)*}$ of $J_{k-\frac{1}{2},m}^{(n)}$.

Let $M_{k-\frac{1}{2}}^+(\Gamma_0^{(n+1)}(4))$ be the generalized plus-space introduced in [Ib 92, page 112], which is a generalization of the Kohnen plus-space for higher degrees:

$$M_{k-\frac{1}{2}}^+(\Gamma_0^{(n+1)}(4)) := \left\{ F \in M_{k-\frac{1}{2}}(\Gamma_0^{(n+1)}(4)) \left| \begin{array}{l} \text{the coefficients } A(N) = 0 \text{ unless} \\ N + (-1)^k R^t R \in 4 \operatorname{Sym}_{n+1}^* \\ \text{for some } R \in \mathbb{Z}^{(n+1,1)} \end{array} \right. \right\}.$$

For any even integer k , the isomorphism between $J_{k,1}^{(n+1)}$, which is the space of Jacobi forms of index 1, and $M_{k-\frac{1}{2}}^+(\Gamma_0^{(n+1)}(4))$ is shown in [Ib 92, Theorem 1]. We denote this linear map by σ which is a bijection from $J_{k,1}^{(n+1)}$ to $M_{k-\frac{1}{2}}^+(\Gamma_0^{(n+1)}(4))$ as modules over the ring of Hecke operators. The map σ is given via

$$\begin{aligned} & \sum_{\substack{N \in \operatorname{Sym}_n^*, R \in \mathbb{Z}^{(n,1)} \\ 4N - R^t R \geq 0}} C(N, R) e(N\tau + R^t z) \\ \mapsto & \sum_{\substack{R \bmod (2\mathbb{Z})^{(n,1)} \\ R \in \mathbb{Z}^{(n,1)}}} \sum_{\substack{N \in \operatorname{Sym}_n^* \\ 4N - R^t R \geq 0}} C(N, R) e((4N - R^t R)\tau). \end{aligned}$$

The $\mathbb{C}$ -vector subspace $M_{k-\frac{1}{2}}^*(\Gamma_0^{(n+1)}(4))$ of $M_{k-\frac{1}{2}}^+(\Gamma_0^{(n+1)}(4))$ denotes the image of $J_{k,1}^{(n+1)*}$ by the map σ , where $J_{k,1}^{(n+1)*}$ was defined in §4.1.

Let $E_{k,1}^{(n+1)}$ be the first Fourier-Jacobi coefficient of Siegel-Eisenstein series $E_k^{(n+2)}$. It is known that $E_{k,1}^{(n+1)}$ coincides the Jacobi-Eisenstein series of weight k of index 1 of degree $n+1$ (cf. [Bo 83, Satz 7].) We define the *generalized Cohen-Eisenstein series* $\mathcal{H}_{k-\frac{1}{2}}^{(n+1)}$ of weight $k - \frac{1}{2}$ of degree $n+1$ by

$$\mathcal{H}_{k-\frac{1}{2}}^{(n+1)} := \sigma(E_{k,1}^{(n+1)}).$$

Because $E_{k,1}^{(n+1)} \in J_{k,1}^{(n+1)*}$, we have $\mathcal{H}_{k-\frac{1}{2}}^{(n+1)} \in M_{k-\frac{1}{2}}^*(\Gamma_0^{(n+1)}(4))$.

For any integer m we denote by $\widetilde{\operatorname{FJ}}_m$ the linear map from $M_{k-\frac{1}{2}}(\Gamma_0^{(n+1)}(4))$ to $J_{k-\frac{1}{2},m}^{(n)}$ obtained by the Fourier-Jacobi expansion with respect to the index m .

We denote by $J_{k-\frac{1}{2},m}^{(n)*}$ the image of $M_{k-\frac{1}{2}}^*(\Gamma_0^{(n+1)}(4))$ through the map $\widetilde{\operatorname{FJ}}_m$.

We recall that the form $e_{k,m}^{(n)}$ was defined as the m -th Fourier-Jacobi coefficient of the generalized Cohen-Eisenstein series $\mathcal{H}_{k-\frac{1}{2}}^{(n+1)}$ (cf. §1). Thus $e_{k,m}^{(n)} \in J_{k-\frac{1}{2},m}^{(n)*}$.

4.4. The map $\iota_{\mathcal{M}}$. We recall $\mathcal{M} = \begin{pmatrix} l & \frac{r}{2} \\ \frac{r}{2} & 1 \end{pmatrix} \in \operatorname{Sym}_2^+$. In this subsection we shall introduce a map

$$\iota_{\mathcal{M}} : H_{\mathcal{M}}^{(n)} \rightarrow \operatorname{Hol}(\mathfrak{H}_n \times \mathbb{C}^{(n,1)} \rightarrow \mathbb{C}),$$

where $H_{\mathcal{M}}^{(n)}$ is a certain subspace of holomorphic functions on $\mathfrak{H}_n \times \mathbb{C}^{(n,2)}$, which will be defined below, and where $\text{Hol}(\mathfrak{H}_n \times \mathbb{C}^{(n,1)} \rightarrow \mathbb{C})$ denotes the space of all holomorphic functions on $\mathfrak{H}_n \times \mathbb{C}^{(n,1)}$. We will show that the restriction of $\iota_{\mathcal{M}}$ gives a linear isomorphism between $J_{k,\mathcal{M}}^{(n)*}$ and $J_{k-\frac{1}{2},m}^{(n)*}$ (cf. Lemma 4.2.)

Let ϕ be a holomorphic function on $\mathfrak{H}_n \times \mathbb{C}^{(n,2)}$. We assume that ϕ has a Fourier expansion

$$\phi(\tau, z) = \sum_{\substack{N \in \text{Sym}_n^*, R \in \mathbb{Z}^{(n,1)} \\ 4N - R\mathcal{M}^{-1}R \geq 0}} A(N, R) e(N\tau + {}^t Rz)$$

for $(\tau, z) \in \mathfrak{H}_n \times \mathbb{C}^{(n,2)}$, and assume that ϕ satisfies the following condition on the Fourier coefficients: if

$$\begin{pmatrix} N & \frac{1}{2}R \\ \frac{1}{2}{}^t R & \mathcal{M} \end{pmatrix} = \begin{pmatrix} N' & \frac{1}{2}R' \\ \frac{1}{2}{}^t R' & \mathcal{M} \end{pmatrix} \begin{bmatrix} 1_n & \\ {}^t T & 1_2 \end{bmatrix}$$

with some $T = (0, \lambda) \in \mathbb{Z}^{(n,2)}$, $\lambda \in \mathbb{Z}^{(n,1)}$, then $A(N, R) = A(N', R')$.

The symbol $H_{\mathcal{M}}^{(n)}$ denotes the $\mathbb{C}$ -vector space consists of all holomorphic functions which satisfy the above condition.

We remark $J_{k,\mathcal{M}}^{(n)*} \subset J_{k,\mathcal{M}}^{(n)} \subset H_{\mathcal{M}}^{(n)}$ for any even integer k .

Now we shall define a map $\iota_{\mathcal{M}}$. For $\phi(\tau', z') = \sum A(N, R) e(N\tau' + R^t z') \in H_{\mathcal{M}}^{(n)}$ we define a holomorphic function $\iota_{\mathcal{M}}(\phi)$ on $\mathfrak{H}_n \times \mathbb{C}^{(n,1)}$ by

$$\iota_{\mathcal{M}}(\phi)(\tau, z) := \sum_{\substack{M \in \text{Sym}_n^*, S \in \mathbb{Z}^{(n,1)} \\ 4Mm - S^t S \geq 0}} C(M, S) e(M\tau + S^t z),$$

for $(\tau, z) \in \mathfrak{H}_n \times \mathbb{C}^{(n,1)}$, where we define $C(M, S) := A(N, R)$ if there exists matrices $N \in \text{Sym}_2^*$ and $R = (R_1, R_2) \in \mathbb{Z}^{(n,2)}$ ($R_1, R_2 \in \mathbb{Z}^{(n,1)}$) which satisfy

$$\begin{pmatrix} M & \frac{1}{2}S \\ \frac{1}{2}{}^t S & \det(2\mathcal{M}) \end{pmatrix} = 4 \begin{pmatrix} N & \frac{1}{2}R_1 \\ \frac{1}{2}{}^t R_1 & l \end{pmatrix} - \begin{pmatrix} R_2 \\ r \end{pmatrix} ({}^t R_2, r),$$

$C(M, S) := 0$ otherwise. We remark that the coefficient $C(M, S)$ does not depend on the choice of the matrices N and R , because if

$$4 \begin{pmatrix} N & \frac{1}{2}R_1 \\ \frac{1}{2}{}^t R_1 & l \end{pmatrix} - \begin{pmatrix} R_2 \\ r \end{pmatrix} ({}^t R_2, r) = 4 \begin{pmatrix} N' & \frac{1}{2}R'_1 \\ \frac{1}{2}{}^t R'_1 & l \end{pmatrix} - \begin{pmatrix} R'_2 \\ r \end{pmatrix} ({}^t R'_2, r),$$

then $4N - R_2^t R_2 = 4N' - R_2'^t R_2'$. Hence $R_2^t R_2 \equiv R_2'^t R_2' \pmod{4}$. Thus there exists a matrix $\lambda \in \mathbb{Z}^{(n,1)}$ such that $R_2' = R_2 + 2\lambda$. Therefore, by straightforward calculation we have

$$\begin{pmatrix} N & \frac{1}{2}R \\ \frac{1}{2}{}^t R & \mathcal{M} \end{pmatrix} = \begin{pmatrix} N' & \frac{1}{2}R' \\ \frac{1}{2}{}^t R' & \mathcal{M} \end{pmatrix} \begin{bmatrix} 1_n & 0 \\ {}^t T & 1_2 \end{bmatrix}$$

with $T = (0, \lambda)$, $R = (R_1, R_2)$ and $R' = (R'_1, R'_2)$. Because ϕ belongs to $H_{\mathcal{M}}^{(n)}$, the above definition of $C(M, S)$ is well-defined.

Now, we restrict the above map $\iota_{\mathcal{M}}$ to the subspace $J_{k,1}^{(n+1)*} \subset J_{k,1}^{(n+1)}$, then we obtain the following lemma.

Lemma 4.2. *Let k be an even integer. We put $m = \det(2\mathcal{M})$. Then we have the following commutative diagram:*

$$\begin{array}{ccc} J_{k,1}^{(n+1)*} & \xrightarrow{\sigma|_{J_{k,1}^{(n+1)*}}} & M_{k-\frac{1}{2}}^*(\Gamma_0^{(n+1)}(4)) \\ FJ_{1,\mathcal{M}}|_{J_{k,1}^{(n)*}} \downarrow & & \downarrow \widetilde{FJ}_m|_{M_{k-\frac{1}{2}}^*(\Gamma_0^{(n)}(4))} \\ J_{k,\mathcal{M}}^{(n)*} & \xrightarrow{\iota_{\mathcal{M}}|_{J_{k,\mathcal{M}}^{(n)*}}} & J_{k-\frac{1}{2},m}^{(n)*}. \end{array}$$

Moreover, the restriction of the linear map $\iota_{\mathcal{M}}$ on $J_{k,\mathcal{M}}^{(n)*}$ is a bijection between $J_{k,\mathcal{M}}^{(n)*}$ and $J_{k-\frac{1}{2},m}^{(n)*}$.

Proof. Let $\psi \in J_{k,1}^{(n+1)*}$. Due to the definition of σ (cf. §4.3) and $\iota_{\mathcal{M}}$, it is not difficult to see $\iota_{\mathcal{M}}(FJ_{1,\mathcal{M}}(\psi)) = \widetilde{FJ}_m(\sigma(\psi))$. Namely, we have the above commutative diagram.

Because the map $\widetilde{FJ}_m|_{M_{k-\frac{1}{2}}^*(\Gamma_0^{(n+1)}(4))} : M_{k-\frac{1}{2}}^*(\Gamma_0^{(n+1)}(4)) \rightarrow J_{k-\frac{1}{2},m}^{(n)*}$ is surjective and because σ is an isomorphism, the map $\iota_{\mathcal{M}}|_{J_{k,\mathcal{M}}^{(n)*}} : J_{k,\mathcal{M}}^{(n)*} \rightarrow J_{k-\frac{1}{2},m}^{(n)*}$ is surjective. The injectivity of the map $\iota_{\mathcal{M}}|_{J_{k,\mathcal{M}}^{(n)*}} : J_{k,\mathcal{M}}^{(n)*} \rightarrow J_{k-\frac{1}{2},m}^{(n)*}$ follows directly from the definition of the map $\iota_{\mathcal{M}}$. $\square$

4.5. Compatibility between index-shift maps and $\iota_{\mathcal{M}}$. In this subsection we shall show compatibility between some index-shift maps and the map $\iota_{\mathcal{M}}$.

For function ψ on $\mathfrak{H}_n \times \mathbb{C}^{(n,2)}$ and for $L \in \mathbb{Z}^{(2,2)}$ we define the function $\psi|U_L$ on $\mathfrak{H}_n \times \mathbb{C}^{(n,2)}$ by

$$(\psi|U_L)(\tau, z) := \psi(\tau, z^t L).$$

For function ϕ on $\mathfrak{H}_n \times \mathbb{C}^{(n,1)}$ and for integer a we define the function $\phi|U_a$ on $\mathfrak{H}_n \times \mathbb{C}^{(n,1)}$ by

$$(\phi|U_a)(\tau, z) := \phi(\tau, az).$$

Proposition 4.3. *For any $\phi \in J_{k,\mathcal{M}}^{(n)*}$ and for any $L = \begin{pmatrix} a & \\ & 1 \end{pmatrix} \in \mathbb{Z}^{(2,2)}$ we obtain*

$$\iota_{\mathcal{M}[L]}(\phi|U_L) = \iota_{\mathcal{M}}(\phi)|U_a.$$

In particular, for any prime p we have $\iota_{\mathcal{M}[\begin{pmatrix} p & \\ & 1 \end{pmatrix}]}(\phi|U_{\begin{pmatrix} p & \\ & 1 \end{pmatrix}}) = \iota_{\mathcal{M}}(\phi)|U_p$.

Proof. We put $m = \det(2\mathcal{M})$. Let $\phi(\tau, z') = \sum_{\substack{N \in \text{Sym}_n^*, R \in \mathbb{Z}^{(n,2)} \\ 4N - R\mathcal{M}^{-1t}R \geq 0}} A(N, R)e(N\tau + R^t z')$ be a Fourier expansion of ϕ . And let

$$\begin{aligned} \iota_{\mathcal{M}}(\phi)(\tau, z) &= \sum_{\substack{M \in \text{Sym}_n^*, S \in \mathbb{Z}^{(n,1)} \\ 4Mm - S^t S \geq 0}} C(M, S) e(M\tau + S^t z), \\ \iota_{\mathcal{M}[L]}(\phi|U_L)(\tau, z) &= \sum_{\substack{M \in \text{Sym}_n^*, S \in \mathbb{Z}^{(n,1)} \\ 4Mma^2 - S^t S \geq 0}} C_1(M, S) e(M\tau + S^t z) \end{aligned}$$

and

$$(\iota_{\mathcal{M}}(\phi)|U_a)(\tau, z) = \sum_{\substack{M \in \text{Sym}_n^*, S \in \mathbb{Z}^{(n,1)} \\ 4Mma^2 - S^t S \geq 0}} C_2(M, S) e(M\tau + S^t z)$$

be Fourier expansions.

We have $C_2(M, S) = C(M, a^{-1}S)$. Moreover, we obtain $C_1(M, S) = A(N, RL^{-1})$ with some $N \in \text{Sym}_n^*$ and $R \in \mathbb{Z}^{(n,2)}$ such that

$$\begin{pmatrix} M & \frac{1}{2}S \\ \frac{1}{2}S & ma^2 \end{pmatrix} = 4 \begin{pmatrix} N & \frac{1}{2}R \\ \frac{1}{2}R & \mathcal{M}[L] \end{pmatrix} \begin{bmatrix} \begin{pmatrix} 0 \\ 1_n \\ 0 \\ 0 \dots 0 \\ -\frac{1}{2}{}^t(R \begin{pmatrix} 0 \\ 1 \end{pmatrix}) \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ 1 \\ -\frac{1}{2}ra - b \end{pmatrix} \end{bmatrix}.$$

With the above matrices N , R , M and S we have

$$\begin{aligned} \begin{pmatrix} M & \frac{1}{2}a^{-1}S \\ \frac{1}{2}a^{-1}S & m \end{pmatrix} &= 4 \begin{pmatrix} N & \frac{1}{2}R \\ \frac{1}{2}R & \mathcal{M}[L] \end{pmatrix} \begin{bmatrix} \begin{pmatrix} 0 \\ 1_n \\ 0 \\ 0 \dots 0 \\ -\frac{1}{2}{}^t(R \begin{pmatrix} 0 \\ 1 \end{pmatrix}) \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1_n \\ 0 \\ 0 \dots 0 \\ a^{-1} \end{pmatrix} \end{bmatrix} \\ &= 4 \begin{pmatrix} N & \frac{1}{2}RL^{-1} \\ \frac{1}{2}{}^t(RL^{-1}) & \mathcal{M} \end{pmatrix} \begin{bmatrix} \begin{pmatrix} 0 \\ 1_{n+1} \\ 0 \\ -\frac{1}{2}{}^t(RL^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}) \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{2}r \end{pmatrix} \end{bmatrix}. \end{aligned}$$

Thus $C_2(M, S) = C(M, a^{-1}S) = A(N, RL^{-1}) = C_1(M, S)$. $\square$

Proposition 4.4. *For odd prime p and for $0 \leq \alpha \leq n$, let $\tilde{V}_{\alpha, n-\alpha}(p^2)$ and $V_{\alpha, n-\alpha}(p^2)$ be index-shift maps defined in §2.7. Then, for any $\phi \in J_{k, \mathcal{M}}^{(n)*}$ we have*

$$(4.1) \quad \iota_{\mathcal{M}}(\phi)|\tilde{V}_{\alpha, n-\alpha}(p^2) = p^{k(2n+1)-n(n+\frac{7}{2})+\frac{1}{2}\alpha} \iota_{\mathcal{M}[\begin{pmatrix} p & \\ & 1 \end{pmatrix}]}(\phi|V_{\alpha, n-\alpha}(p^2)).$$

Proof. We compare the Fourier coefficients of the both sides of (4.1). Let

$$\begin{aligned}\phi(\tau, z') &= \sum_{N,R} A_1(N, R) e(N\tau + R^t z'), \\ (\phi|V_{\alpha, n-\alpha}(p^2))(\tau, z') &= \sum_{N,R} A_2(N, R) e(N\tau + R^t z'), \\ (\iota_{\mathcal{M}}(\phi))(\tau, z) &= \sum_{M,S} C_1(M, S) e(M\tau + S^t z)\end{aligned}$$

and

$$(\iota_{\mathcal{M}}(\phi)|\tilde{V}_{\alpha, n-\alpha}(p^2))(\tau, z) = \sum_{M,S} C_2(M, S) e(M\tau + S^t z)$$

be Fourier expansions, where $\tau \in \mathfrak{H}_n$, $z' \in \mathbb{C}^{(n,2)}$ and $z \in \mathbb{C}^{(n,1)}$. For the sake of simplicity we put $U = \begin{pmatrix} p^2 & \\ & p \end{pmatrix}$. Then

$$\begin{aligned}& \phi|V_{\alpha, n-\alpha}(p^2) \\ &= \sum_{\begin{pmatrix} p^{2t}D^{-1} & B \\ 0_n & D \end{pmatrix}} \sum_{\lambda_2, \mu_2 \in (\mathbb{Z}/p\mathbb{Z})^{(n,1)}} \\ & \quad \times \phi|_{k, \mathcal{M}} \left(\begin{pmatrix} p^{2t}D^{-1} & B \\ 0_n & D \end{pmatrix} \times \begin{pmatrix} U & \\ & p^2U^{-1} \end{pmatrix}, [((0, \lambda_2), (0, \mu_2)), 0_2] \right) \\ &= \sum_{\begin{pmatrix} p^{2t}D^{-1} & B \\ 0_n & D \end{pmatrix}} \sum_{\lambda_2, \mu_2 \in (\mathbb{Z}/p\mathbb{Z})^{(n,1)}} \sum_{N,R} A(N, R) \\ & \quad \times e(N\tau + R^t z)|_{k, \mathcal{M}} \left(\begin{pmatrix} p^{2t}D^{-1} & B \\ 0_n & D \end{pmatrix} \times \begin{pmatrix} U & \\ & p^2U^{-1} \end{pmatrix}, [((0, \lambda_2), (0, \mu_2)), 0_2] \right),\end{aligned}$$

where, in the above summations, $\begin{pmatrix} p^{2t}D^{-1} & B \\ 0_n & D \end{pmatrix}$ run over a set of all representatives of $\Gamma_n \backslash \Gamma_n \text{diag}(1_\alpha, p1_{n-\alpha}, p^2 1_\alpha, p1_{n-\alpha}) \Gamma_n$, and where the slash operator $|_{k, \mathcal{M}}$ is defined in §2.4.

We put $\lambda = (0, \lambda_2)$, $\mu = (0, \mu_2) \in \mathbb{Z}^{(n,2)}$, then we obtain

$$\begin{aligned}& e(N\tau + R^t z)|_{k, \mathcal{M}} \left(\begin{pmatrix} p^{2t}D^{-1} & B \\ 0_n & D \end{pmatrix} \times \begin{pmatrix} U & \\ & p^2U^{-1} \end{pmatrix}, [(\lambda, \mu), 0_2] \right) \\ &= p^{-k} \det(D)^{-k} e(\hat{N}\tau + \hat{R}^t z + NBD^{-1} + RU^t \mu D^{-1}),\end{aligned}$$

where

$$\hat{N} = p^2 D^{-1} N^t D^{-1} + D^{-1} R U^t \lambda + \frac{1}{p^2} \lambda U M U^t \lambda$$

and

$$\hat{R} = D^{-1} R U + \frac{2}{p^2} \lambda U M U.$$

Thus

$$N = \frac{1}{p^2} D \left(\left(\hat{N} - \frac{1}{4} \hat{R}_2^t \hat{R}_2 \right) + \frac{1}{4} (\hat{R}_2 - 2\lambda_2)^t (\hat{R}_2 - 2\lambda_2) \right)^t D$$

and

$$R = D \left(\hat{R} - \frac{2}{p^2} \lambda U \mathcal{M} U \right) U^{-1},$$

where $\hat{R}_2 = \hat{R} \begin{pmatrix} 0 & \\ & 1 \end{pmatrix}$.

Hence, for any $\hat{N} \in \text{Sym}_n^*$ and for any $\hat{R} \in \mathbb{Z}^{(n,2)}$ we have

$$\begin{aligned} & A_2(\hat{N}, \hat{R}) \\ &= p^{-k} \sum_{\begin{pmatrix} p^{2t} D^{-1} & B \\ 0_n & D \end{pmatrix}} \det(D)^{-k} \sum_{\lambda_2 \in (\mathbb{Z}/p\mathbb{Z})^{(n,1)}} \sum_{\mu_2 \in (\mathbb{Z}/p\mathbb{Z})^{(n,1)}} A_1(N, R) e(NBD^{-1} + RU^t(0, \mu_2)D^{-1}) \\ &= p^{-k+n} \sum_{\begin{pmatrix} p^{2t} D^{-1} & B \\ 0_n & D \end{pmatrix}} \det(D)^{-k} \sum_{\lambda_2 \in (\mathbb{Z}/p\mathbb{Z})^{(n,1)}} A_1(N, R) e(NBD^{-1}), \end{aligned}$$

where N and R are the same symbols as the above, which are determined by $\hat{N}$, $\hat{R}$ and λ_2 , and where, in the above summations, $\begin{pmatrix} p^{2t} D^{-1} & B \\ 0_n & D \end{pmatrix}$ runs over a complete set of representatives of $\Gamma_n \backslash \Gamma_n \text{diag}(1_\alpha, p1_{n-\alpha}, p^2 1_\alpha, p1_{n-\alpha}) \Gamma_n$. We remark that $A_1(N, R) = 0$ unless $N \in \text{Sym}_n^*$ and $R \in \mathbb{Z}^{(n,2)}$.

Due to the definition of $\iota_{\mathcal{M}}$, for $N \in \text{Sym}_n^*$ and $R \in \mathbb{Z}^{(n,2)}$ we have the identity

$$A_1(N, R) = C_1(4N - R \begin{pmatrix} 0 & \\ & 1 \end{pmatrix}^t (R \begin{pmatrix} 0 & \\ & 1 \end{pmatrix}), 4R \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} - 2rR \begin{pmatrix} 0 & \\ & 1 \end{pmatrix}).$$

Here

$$4N - R \begin{pmatrix} 0 & \\ & 1 \end{pmatrix}^t (R \begin{pmatrix} 0 & \\ & 1 \end{pmatrix}) = \frac{1}{p^2} D \left(4\hat{N} - \hat{R}_2^t \hat{R}_2 \right)^t D$$

and

$$4R \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} - 2rR \begin{pmatrix} 0 & \\ & 1 \end{pmatrix} = \frac{1}{p^2} D(4\hat{R} \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} - 2rp\hat{R}_2).$$

Hence we have

$$\begin{aligned} (4.2) \quad & A_2(\hat{N}, \hat{R}) \\ &= p^{-k+n} \sum_{\begin{pmatrix} p^{2t} D^{-1} & B \\ 0_n & D \end{pmatrix}} \det(D)^{-k} C_1 \left(\frac{1}{p^2} D \left(4\hat{N} - \hat{R}_2^t \hat{R}_2 \right)^t D, \frac{1}{p^2} D(4\hat{R} \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} - 2rp\hat{R}_2) \right) \\ & \quad \times e \left(\frac{1}{p^2} \left(\hat{N} - \frac{1}{4} \hat{R}_2^t \hat{R}_2 \right)^t DB \right) \sum_{\lambda_2} e \left(\frac{1}{4p^2} (\hat{R}_2 - 2\lambda_2)^t (\hat{R}_2 - 2\lambda_2)^t DB \right), \end{aligned}$$

where λ_2 runs over a complete set of representatives of $(\mathbb{Z}/p\mathbb{Z})^{(n,1)}$ such that

$$D \left(\hat{R} - \frac{2}{p^2}(0, \lambda_2)U\mathcal{M}U \right) U^{-1} \in \mathbb{Z}^{(n,2)}.$$

Let $\mathfrak{S}_\alpha$ be a complete set of representative of $\Gamma_n \backslash \Gamma_n \begin{pmatrix} 1_\alpha & & & \\ & p1_{n-\alpha} & & \\ & & p^2 1_\alpha & \\ & & & p1_{n-\alpha} \end{pmatrix} \Gamma_n$. Now we quote a complete set of representatives $\mathfrak{S}_\alpha$ from [Zh 84]. We put

$$\delta_{i,j} := \text{diag}(1_i, p1_{j-i}, p^2 1_{n-j})$$

and

$$\mathfrak{S}_\alpha := \left\{ \begin{pmatrix} p^2 \delta_{i,j}^{-1} & b_0 \\ 0_n & \delta_{i,j} \end{pmatrix} \begin{pmatrix} {}^t u^{-1} & 0_n \\ 0_n & u \end{pmatrix} \middle| i, j, b_0, u \right\},$$

where, in the above set, i and j run over all non-negative integers such that $j - i - n + \alpha \geq 0$, and where u runs over a complete set of representatives of $(\delta_{i,j}^{-1} \text{GL}_n(\mathbb{Z}) \delta_{i,j} \cap \text{GL}_n(\mathbb{Z})) \backslash \text{GL}_n(\mathbb{Z})$, and b_0 runs over all matrices in the set

$$\mathfrak{T} := \left\{ \begin{pmatrix} 0_i & 0 & 0 \\ 0 & a_1 & pb_1 \\ 0 & {}^t b_1 & b_2 \end{pmatrix} \middle| \begin{array}{l} b_1 \in (\mathbb{Z}/p\mathbb{Z})^{(j-i, n-j)}, b_2 = {}^t b_2 \in (\mathbb{Z}/p^2\mathbb{Z})^{(n-j, n-j)}, \\ a_1 = {}^t a_1 \in (\mathbb{Z}/p\mathbb{Z})^{(j-i, j-i)}, \text{rank}_p(a_1) = j - i - n + \alpha \end{array} \right\}.$$

For a matrix $g = \begin{pmatrix} p^{2t} D^{-1} & B \\ 0_n & D \end{pmatrix} = \begin{pmatrix} p^2 \delta_{i,j}^{-1} & b_0 \\ 0_n & \delta_{i,j} \end{pmatrix} \begin{pmatrix} {}^t u^{-1} & 0_n \\ 0_n & u \end{pmatrix} \in \mathfrak{S}_\alpha$ with a matrix $b_0 = \begin{pmatrix} 0_i & 0 & 0 \\ 0 & a_1 & pb_1 \\ 0 & {}^t b_1 & b_2 \end{pmatrix} \in \mathfrak{T}$, we define $\varepsilon(g) := \left(\frac{-4}{p} \right)^{\text{rank}_p(a_1)/2} \left(\frac{\det a'_1}{p} \right)$, where $a'_1 \in \text{GL}_{j-i-n+\alpha}(\mathbb{Z}/p\mathbb{Z})$ is a matrix, such that $a_1 \equiv \begin{pmatrix} a'_1 & 0 \\ 0 & 0_{n-\alpha} \end{pmatrix} [v] \pmod{p}$ with some $v \in \text{GL}_{j-i}(\mathbb{Z})$. Under the assumption

$$\frac{1}{p^2} D(4\hat{R} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 2rp\hat{R}_2) \in \mathbb{Z}^{(n,2)}$$

the condition $D(\hat{R} - p^{-2}(0, \lambda_2)U\mathcal{M}U)U^{-1} \in \mathbb{Z}^{(n,2)}$ is equivalent to the condition

$$u(\hat{R}_2 - 2\lambda_2) \in \begin{pmatrix} p1_i & 0 \\ 0 & 1_{n-i} \end{pmatrix} \mathbb{Z}^{(n,1)}.$$

Hence the last summation in (4.2) is

$$\begin{aligned}
& \sum_{\lambda_2} e\left(\frac{1}{4p^2}(\hat{R}_2 - 2\lambda_2)^t(\hat{R}_2 - 2\lambda_2)^t DB\right) \\
&= p^{n-j} \sum_{\lambda' \in (\mathbb{Z}/p\mathbb{Z})^{(j-i,1)}} e\left(\frac{1}{p} \lambda'^t a_1 \lambda'\right) \\
&= p^{n-i-\text{rank}_p(a_1)} \left(\left(\frac{-4}{p}\right)p\right)^{\text{rank}_p(a_1)/2} \left(\frac{\det a_1'}{p}\right) \\
&= p^{n-i-\frac{\text{rank}_p(a_1)}{2}} \varepsilon(g) \\
&= p^{n+(n-i-j-\alpha)/2} \varepsilon(g).
\end{aligned}$$

Thus (4.2) is

$$\begin{aligned}
A_2(\hat{N}, \hat{R}) &= p^{-k+2n} \sum_g p^{-k(2n-i-j)+(n-i-j-\alpha)/2} \varepsilon(g) e\left(p^{-2} \left(4\hat{N} - \hat{R}_2^t \hat{R}_2\right)^t DB\right) \\
&\quad \times C_1\left(p^{-2} D(4\hat{N} - \hat{R}_2^t \hat{R}_2)^t D, p^{-2} D(4\hat{R} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 2rp\hat{R}_2)\right),
\end{aligned}$$

where $g = \begin{pmatrix} p^{2t} D^{-1} & B \\ 0_n & D \end{pmatrix} = \begin{pmatrix} p^2 \delta_{i,j}^{-1} & b_0 \\ 0_n & \delta_{i,j} \end{pmatrix} \begin{pmatrix} {}^t u^{-1} & 0_n \\ 0_n & u \end{pmatrix}$ runs over all elements in the set $\mathfrak{S}_\alpha$.

Now we shall express $C_2(M, S)$ as a linear combination of Fourier coefficients $C_1(M, S)$ of $\iota_M(\phi)$. For $Y = (\text{diag}(1_\alpha, p1_{n-\alpha}, p^2 1_\alpha, p1_{n-\alpha}), p^{\alpha/2}) \in \widetilde{\text{GSp}}_n^+(\mathbb{Z})$ a complete set of representatives of $\Gamma_0^{(n)}(4)^* \backslash \Gamma_0^{(n)}(4)^* Y \Gamma_0^{(n)}(4)^*$ is given by elements

$$\tilde{g} = (g, \varepsilon(g) p^{(n-i-j)/2}) \in \widetilde{\text{GSp}}_n^+(\mathbb{Z}),$$

where g runs over all elements in the set $\mathfrak{S}_\alpha$, and $\varepsilon(g)$ is defined as the above (cf. [Zh 84, Lemma 3.2]). Hence

$$\begin{aligned}
& (\iota_M(\phi) | \tilde{V}_{\alpha, n-\alpha}(p^2))(\tau, z) \\
&= p^{n(2k-1)/2-n(n+1)} \sum_{M, S} \sum_{\tilde{g}} p^{(-k+1/2)(n-i-j)} \varepsilon(g) C_1(M, S) \\
&\quad \times e(M(p^{2t} D^{-1} \tau + B) D^{-1} + p^2 S^t z D^{-1}) \\
&= p^{n(2k-1)/2-n(n+1)} \sum_{\hat{M}, \hat{S}} \sum_{g \in \mathfrak{S}_\alpha} p^{(-k+1/2)(n-i-j)} \varepsilon(g) C_1(p^{-2} D \hat{M}^t D, p^{-2} D \hat{S}) \\
&\quad \times e(\hat{M} \tau + \hat{S}^t z + p^{-2} \hat{M}^t DB).
\end{aligned}$$

Thus

$$C_2(\hat{M}, \hat{S}) = \sum_g p^{-n(n+1)+(k-1/2)(i+j)} \varepsilon(g) C_1(p^{-2} D \hat{M}^t D, p^{-2} D \hat{S}) e(p^{-2} \hat{M}^t D B).$$

Now we put $\hat{M} = 4\hat{N} - \hat{R}_2^t \hat{R}_2$ and $\hat{S} = 4\hat{R} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 2rp\hat{R}_2$, then

$$C_2(4\hat{N} - \hat{R}_2^t \hat{R}_2, 4\hat{R} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 2rp\hat{R}_2) = p^{2nk+k-n^2-\frac{7}{2}n+\frac{1}{2}\alpha} A_2(\hat{N}, \hat{R}).$$

The proposition follows from this identity. $\square$

4.6. Index-shift maps at $p = 2$. For $p = 2$ we define the index-shift map $\tilde{T}_{\alpha, n-\alpha}(p^2)$ on $J_{k-\frac{1}{2}, m}^{(n)*}$ through the identity 4.1, namely we define

$$\phi | \tilde{V}_{\alpha, n-\alpha}(4) := 2^{k(2n+1)-n(n+\frac{7}{2})+\frac{1}{2}\alpha} \iota_{\mathcal{M}[\begin{pmatrix} 2 & \\ & 1 \end{pmatrix}]}(\psi | V_{\alpha, n-\alpha}(4))$$

for any $\phi \in J_{k-\frac{1}{2}, m}^{(n)*}$, and where $\psi \in J_{k, \mathcal{M}}^{(n)*}$ is the Jacobi form which satisfies $\iota_{\mathcal{M}}(\psi) = \phi$. We have $\phi | \tilde{V}_{\alpha, n-\alpha}(4) \in J_{k-\frac{1}{2}, 4m}^{(n)}$. The reader is referred to §4.4 for the definition of $\iota_{\mathcal{M}}$ and is referred to §2.7 for the definition of $V_{\alpha, n-\alpha}(4)$.

4.7. Index-shift maps $V_p^{(1)}$, $V_{1,p}^{(2)}$ and $V_{2,p}^{(2)}$. In the case $n = 1, 2$, we write simply

$$\begin{aligned} \varphi | V_p^{(1)} &:= \varphi | \tilde{V}_{1,0}(p^2), \\ \phi | V_{1,p}^{(2)} &:= p^{-k+\frac{7}{2}} \phi | \tilde{V}_{1,1}(p^2), \\ \phi | V_{2,p}^{(2)} &:= \phi | \tilde{V}_{2,0}(p^2) \end{aligned}$$

for any $\varphi \in J_{k-\frac{1}{2}, m}^{(1)*}$, $\phi \in J_{k-\frac{1}{2}, m}^{(2)*}$ and for any prime p . The reader is referred to §4.3 for the definition of the subspace $J_{k-\frac{1}{2}, m}^{(n)*}$ of $J_{k-\frac{1}{2}, m}^{(n)}$, and is referred to §2.7 and §4.6 for the definition of the index-shift maps $\tilde{V}_{1,0}(p^2)$, $\tilde{V}_{1,1}(p^2)$ and $\tilde{V}_{2,0}(p^2)$. We remark that $\varphi | V_p^{(1)} \in J_{k-\frac{1}{2}, mp^2}^{(1)}$ and $\phi | V_{i,p}^{(2)} \in J_{k-\frac{1}{2}, mp^2}^{(2)}$ ($i = 1, 2$).

5. ACTION OF THE INDEX-SHIFT MAPS ON JACOBI-EISENSTEIN SERIES

In this section we fix a positive definite half-integral symmetric matrix $\mathcal{M} = \begin{pmatrix} * & * \\ * & 1 \end{pmatrix} \in \text{Sym}_2^+$. The purpose of this section is to express the function $E_{k, \mathcal{M}}^{(n)} | V_{\alpha, n-\alpha}(p^2)$ as a summation of certain exponential functions with generalized Gauss sums, where $E_{k, \mathcal{M}}^{(n)}$ is the Jacobi-Eisenstein series of index $\mathcal{M}$ (cf. §3), and where $V_{\alpha, n-\alpha}(p^2)$ is an index-shift map (cf. §2.7). We remark that $E_{k, \mathcal{M}}^{(n)} | V_{\alpha, n-\alpha}(p^2) \in J_{k, \mathcal{M}[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}]}^{(n)}$ for any $0 \leq \alpha \leq n$.

First we will express $E_{k, \mathcal{M}}^{(n)} | V_{\alpha, n-\alpha}(p^2)$ as a summation of certain functions $K_{i,j}^\beta$ (cf. Lemma 5.2), and after that, we will express $E_{k, \mathcal{M}}^{(n)} | V_{\alpha, n-\alpha}(p^2)$ as a summation of certain functions $\tilde{K}_{i,j}^\beta$ (cf. Proposition 5.3).

The calculation in this section is an analogue to the one given in [Ya 89] in the case of index 1. However, we need to generalize his calculation for Jacobi-Eisenstein series $E_{k,1}^{(n)}$ of index 1 to our case for $E_{k,\mathcal{M}}^{(n)}$ with $\mathcal{M} = \begin{pmatrix} * & * \\ * & 1 \end{pmatrix} \in \text{Sym}_2^+$. This generalization is not obvious, because we need to treat the action of the Heisenberg group parts $[((0, u_2), (0, v_2)), 0]$, which plays an important rule in this generalization.

5.1. The function $K_{i,j}^\beta$. The purpose of this subsection is to introduce functions $K_{i,j}^\beta$ and to express $E_{k,\mathcal{M}}^{(n)}|V_{\alpha,n-\alpha}(p^2)$ as a summation of $K_{i,j}^\beta$. Moreover, we shall calculate $K_{i,j}^\beta$ more precisely (cf. Lemma 5.2).

We put $\delta_{i,j} := \text{diag}(1_i, p1_{j-i}, p^2 1_{n-j})$. For $x = \text{diag}(0_i, x', 0_{n-i-j})$ with $x' = {}^t x' \in \mathbb{Z}^{(j-i, j-i)}$, we set $\delta_{i,j}(x) := \begin{pmatrix} p^2 \delta_{i,j}^{-1} & x \\ 0 & \delta_{i,j} \end{pmatrix}$ and $\Gamma(\delta_{i,j}(x)) := \Gamma_n \cap \delta_{i,j}(x)^{-1} \Gamma_\infty^{(n)} \delta_{i,j}(x)$.

For $x = \text{diag}(0_i, x', 0_{n-i-j})$, $y = \text{diag}(0_i, y', 0_{n-i-j})$ with $x' = {}^t x'$, $y' = {}^t y' \in \mathbb{Z}^{(j-i, j-i)}$, we say that x and y are *equivalent*, if there exists a matrix $u \in \text{GL}_n(\mathbb{Z}) \cap \delta_{i,j} \text{GL}_n(\mathbb{Z}) \delta_{i,j}^{-1}$ which has a form $u = \begin{pmatrix} u_1 & * & * \\ * & u_2 & * \\ * & * & u_3 \end{pmatrix}$ satisfying $x' \equiv u_2 y' {}^t u_2 \pmod{p}$, where $u_2 \in \mathbb{Z}^{(j-i, j-i)}$, $u_1 \in \mathbb{Z}^{(i, i)}$ and $u_3 \in \mathbb{Z}^{(n-j, n-j)}$.

We denote by $[x]$ the equivalent class of x . We quote the following lemma from [Ya 89].

Lemma 5.1. *The double coset $\Gamma_n \text{diag}(1_\alpha, p1_{n-\alpha}, p^2 1_\alpha, p1_{n-\alpha}) \Gamma_n$ is written as a disjoint union*

$$\Gamma_n \begin{pmatrix} 1_\alpha & & & \\ & p1_{n-\alpha} & & \\ & & p^2 1_\alpha & \\ & & & p1_{n-\alpha} \end{pmatrix} \Gamma_n = \bigcup_{0 \leq i \leq j \leq n} \bigcup_{[x]} \Gamma_\infty^{(n)} \delta_{i,j}(x) \Gamma_n,$$

where $[x]$ runs over all equivalent classes which satisfy $\text{rank}_p(x) = j - i - n + \alpha \geq 0$.

Proof. The reader is referred to [Ya 89, Corollary 2.2]. □

$$\text{We put } U := \begin{pmatrix} p^2 & 0 \\ 0 & p \end{pmatrix}.$$

By the definition of the index-shift map $V_{\alpha, n-\alpha}(p^2)$ and of the Jacobi-Eisenstein series $E_{k, \mathcal{M}}^{(n)}$, we have

$$\begin{aligned}
& E_{k, \mathcal{M}}^{(n)} | V_{\alpha, n-\alpha}(p^2) \\
&= \sum_{u, v \in \mathbb{Z}^{(n,1)}} \sum_{M' \in \Gamma_n \setminus \Gamma_n \text{diag}(1_\alpha, p1_{n-\alpha}, p^2 1_\alpha, p1_{n-\alpha}) \Gamma_n} \sum_{M \in \Gamma_\infty^{(n)} \setminus \Gamma_n} \sum_{\lambda \in \mathbb{Z}^{(n,1)}} \\
& \quad 1|_{k, \mathcal{M}}([(\lambda, 0), 0], MM' \times \begin{pmatrix} U & 0 \\ 0 & p^2 U^{-1} \end{pmatrix})|_{k, \mathcal{M}[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}]} [((0, u), (0, v)), 0] \\
&= \sum_{u, v \in \mathbb{Z}^{(n,1)}} \sum_{M \in \Gamma_\infty^{(n)} \setminus \Gamma_n \text{diag}(1_\alpha, p1_{n-\alpha}, p^2 1_\alpha, p1_{n-\alpha}) \Gamma_n} \sum_{\lambda \in \mathbb{Z}^{(n,1)}} \\
& \quad 1|_{k, \mathcal{M}}([(\lambda, 0), 0], M \times \begin{pmatrix} U & 0 \\ 0 & p^2 U^{-1} \end{pmatrix})|_{k, \mathcal{M}[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}]} [((0, u), (0, v)), 0].
\end{aligned}$$

Hence, due to Lemma 5.1, we have

$$\begin{aligned}
& E_{k, \mathcal{M}}^{(n)} | V_{\alpha, n-\alpha}(p^2) \\
&= \sum_{u, v \in \mathbb{Z}^{(n,1)}} \sum_{\substack{i, j \\ 0 \leq i \leq j \leq n}} \sum_{\substack{[x] \\ \text{rank}_p(x) = j-i-n+\alpha}} \sum_{M \in \Gamma_\infty^{(n)} \setminus \delta_{i,j}(x) \Gamma_n} \sum_{\lambda \in \mathbb{Z}^{(n,1)}} \\
& \quad 1|_{k, \mathcal{M}}([(\lambda, 0), 0], M \times \begin{pmatrix} U & 0 \\ 0 & p^2 U^{-1} \end{pmatrix})|_{k, \mathcal{M}[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}]} [((0, u), (0, v)), 0] \\
&= \sum_{u, v \in \mathbb{Z}^{(n,1)}} \sum_{\substack{i, j \\ 0 \leq i \leq j \leq n}} \sum_{\substack{[x] \\ \text{rank}_p(x) = j-i-n+\alpha}} \sum_{M \in \Gamma(\delta_{i,j}(x)) \setminus \Gamma_n} \sum_{\lambda \in \mathbb{Z}^{(n,1)}} \\
& \quad 1|_{k, \mathcal{M}}([(\lambda, 0), 0], \delta_{i,j}(x) M \times \begin{pmatrix} U & 0 \\ 0 & p^2 U^{-1} \end{pmatrix})|_{k, \mathcal{M}[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}]} [((0, u), (0, v)), 0].
\end{aligned}$$

For $\beta \leq j - i$ we define a function

$$\begin{aligned}
& K_{i,j}^\beta(\tau, z) \\
&:= K_{i,j, \mathcal{M}, p}^\beta(\tau, z) \\
&= \sum_{\substack{[x] \\ \text{rank}_p(x) = \beta}} \sum_{M \in \Gamma(\delta_{i,j}(x)) \setminus \Gamma_n} \sum_{\lambda \in \mathbb{Z}^n} \{1|_{k, \mathcal{M}}([(\lambda, 0), 0], \delta_{i,j}(x) M \times \begin{pmatrix} U & 0 \\ 0 & p^2 U^{-1} \end{pmatrix})\}(\tau, z).
\end{aligned}$$

Then

$$E_{k, \mathcal{M}}^{(n)} | V_{\alpha, n-\alpha}(p^2) = \sum_{\substack{i, j \\ 0 \leq i \leq j \leq n}} \sum_{u, v \in \mathbb{Z}^{(n,1)}} K_{i,j}^{\alpha-i-n+j}(\tau, z)|_{k, \mathcal{M}[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}]} [((0, u), (0, v)), 0].$$

We define

$$L_{i,j} := L_{i,j, \mathcal{M}, p} = \left\{ \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} \mid \begin{array}{l} \lambda_1 \in (p\mathbb{Z})^{(i,2)}, \lambda_2 \in \mathbb{Z}^{(j-i,2)}, \lambda_3 \in (p^{-1}\mathbb{Z})^{(n-j,2)} \\ 2\lambda_2 \mathcal{M}^t \lambda_3 \in \mathbb{Z}^{(j-i, n-j)}, \lambda_3 \mathcal{M}^t \lambda_3 \in \mathbb{Z}^{(n-j, n-j)} \end{array} \right\}.$$

Moreover, we define a subgroup $\Gamma(\delta_{i,j})$ of $\Gamma_\infty^{(n)}$ by

$$\Gamma(\delta_{i,j}) := \left\{ \begin{pmatrix} A & B \\ 0_n & {}^t A^{-1} \end{pmatrix} \in \Gamma_\infty^{(n)} \mid A \in \delta_{i,j} \mathrm{GL}_n(\mathbb{Z}) \delta_{i,j}^{-1} \right\}.$$

Lemma 5.2. *Let $K_{i,j}^\beta$ be as the above. We obtain*

$$\begin{aligned} K_{i,j}^\beta(\tau, z) &= p^{-k(2n-i-j+1)+(n-j)(n-i+1)} \sum_{M \in \Gamma(\delta_{i,j}) \backslash \Gamma_n} \\ &\quad \times \sum_{\lambda \in L_{i,j}} 1|_{k,\mathcal{M}}([(\lambda, 0), 0_n], M)(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) \sum_{\substack{x = {}^t x \in (\mathbb{Z}/p\mathbb{Z})^{(n,n)} \\ x = \mathrm{diag}(0_i, x', 0_{n-j}) \\ \mathrm{rank}_p(x') = \beta}} e \left(\frac{1}{p} \mathcal{M}^t \lambda x \lambda \right), \end{aligned}$$

where, in the last summation, x runs over a complete set of representatives of $(\mathbb{Z}/p\mathbb{Z})^{(n,n)}$ such that $x = {}^t x$, $\mathrm{rank}_p(x) = \beta$ and $x = \mathrm{diag}(0_i, x', 0_{n-j})$ with some $x' \in (\mathbb{Z}/p\mathbb{Z})^{(j-i, j-i)}$.

Proof. We proceed as in [Ya 89, Proposition 3.2]. The inside of the last summation of $K_{i,j}^\beta(\tau, z)$ is

$$\begin{aligned} & \left(1|_{k,\mathcal{M}}([(\lambda, 0), 0], \delta_{i,j}(x)M \times \begin{pmatrix} U & 0 \\ 0 & p^2 U^{-1} \end{pmatrix}) \right) (\tau, z) \\ &= \det(p^2 U^{-1})^{-k} \det(\delta_{i,j})^{-k} \\ &\quad \times \left(e(\mathcal{M}({}^t \lambda (p^2 \delta_{i,j}^{-1} \tau + x) \delta_{i,j}^{-1} \lambda + 2 {}^t \lambda \delta_{i,j}^{-1} z \begin{pmatrix} p^2 & \\ & p \end{pmatrix}))|_{k,\mathcal{M}[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}]} M \right) (\tau, z) \\ &= p^{-k(2n-i-j+1)} \\ &\quad \times \left(\left((1|_{k,\mathcal{M}}([(p \delta_{i,j}^{-1} \lambda, 0), 0], \begin{pmatrix} 1 & p^{-1} x \\ 0 & 1 \end{pmatrix}))(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}))|_{k,\mathcal{M}[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}]} M \right) (\tau, z) \\ &= p^{-k(2n-i-j+1)} \left(1|_{k,\mathcal{M}}([(p \delta_{i,j}^{-1} \lambda, 0), 0], \begin{pmatrix} 1 & p^{-1} x \\ 0 & 1 \end{pmatrix} M) \right) (\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}). \end{aligned}$$

Here we used the identity $\delta_{i,j}x = \delta_{i,j} \mathrm{diag}(0_i, x', 0_{n-j}) = px$. Thus

$$\begin{aligned} K_{i,j}^\beta(\tau, z) &= p^{-k(2n-i-j+1)} \sum_{\substack{[x] \\ \mathrm{rank}_p(x) = \beta}} \sum_{M \in \Gamma(\delta_{i,j}(x)) \backslash \Gamma_n} \\ &\quad \times \sum_{\lambda \in \mathbb{Z}^n} 1|_{k,\mathcal{M}} \left([(p \delta_{i,j}^{-1} \lambda, 0), 0_n], \begin{pmatrix} 1 & p^{-1} x \\ 0 & 1 \end{pmatrix} M \right) (\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}). \end{aligned}$$

We put

$$\mathcal{U} := \left\{ \begin{pmatrix} 1_n & s \\ 0_n & 1_n \end{pmatrix} \mid s = {}^t s \in \mathbb{Z}^{(n,n)} \right\}.$$

Then the set

$$\left\{ \begin{pmatrix} 1_n & s \\ 0_n & 1_n \end{pmatrix} \middle| s = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & s_2 \\ 0 & {}^t s_2 & s_3 \end{pmatrix}, s_2 \in (\mathbb{Z}/p\mathbb{Z})^{(j-i, n-j)}, s_3 = {}^t s_3 \in (\mathbb{Z}/p\mathbb{Z})^{(n-j, n-j)} \right\}$$

is a complete set of representatives of $\Gamma(\delta_{i,j}(x)) \backslash \Gamma(\delta_{i,j}(x))\mathcal{U}$. Therefore

$$\begin{aligned} & K_{i,j}^\beta(\tau, z) \\ = & p^{-k(2n-i-j+1)} \sum_{\substack{[x] \\ \text{rank}_p(x)=\beta}} \sum_{M \in (\Gamma(\delta_{i,j}(x))\mathcal{U}) \backslash \Gamma_n} \sum_{\lambda \in \mathbb{Z}^n} \sum_{\begin{pmatrix} 1_n & s \\ 0 & 1_n \end{pmatrix} \in \Gamma(\delta_{i,j}(x)) \backslash (\Gamma(\delta_{i,j}(x))\mathcal{U})} \\ & \times 1|_{k,\mathcal{M}}([p\delta_{i,j}^{-1}\lambda, 0], 0_n], \begin{pmatrix} 1_n & p^{-1}x \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} 1_n & s \\ 0 & 1_n \end{pmatrix} M)(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) \end{aligned}$$

Hence

$$\begin{aligned} K_{i,j}^\beta(\tau, z) &= p^{-k(2n-i-j+1)} \sum_{\substack{[x] \\ \text{rank}_p(x)=\beta}} \sum_{M \in (\Gamma(\delta_{i,j}(x))\mathcal{U}) \backslash \Gamma_n} \sum_{\lambda \in \mathbb{Z}^n} \\ &\times 1|_{k,\mathcal{M}}([p\delta_{i,j}^{-1}\lambda, 0], 0_n], \begin{pmatrix} 1_n & p^{-1}x \\ 0 & 1_n \end{pmatrix} M)(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) \\ &\times \sum_{\begin{pmatrix} 1_n & s \\ 0 & 1_n \end{pmatrix} \in \Gamma(\delta_{i,j}(x)) \backslash (\Gamma(\delta_{i,j}(x))\mathcal{U})} e(p^2 \mathcal{M}^t \lambda \delta_{i,j}^{-1} s \delta_{i,j}^{-1} \lambda). \end{aligned}$$

The last summation of the above identity is

$$\begin{aligned} & \sum_s e(p^2 \mathcal{M}^t \lambda \delta_{i,j}^{-1} s \delta_{i,j}^{-1} \lambda) \\ = & \begin{cases} p^{(n-j)(n-i+1)} & \text{if } \lambda_3 \mathcal{M}^t \lambda_3 \equiv 0 \pmod{p^2} \text{ and } 2\lambda_3 \mathcal{M}^t \lambda_2 \equiv 0 \pmod{p}, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where $\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} \in \mathbb{Z}^{(n,2)}$ with $\lambda_1 \in \mathbb{Z}^{(i,2)}$, $\lambda_2 \in \mathbb{Z}^{(j-i,2)}$ and $\lambda_3 \in \mathbb{Z}^{(n-j,2)}$.

Thus

$$\begin{aligned} K_{i,j}^\beta(\tau, z) &= p^{-k(2n-i-j+1)+(n-j)(n-i+1)} \sum_{\substack{[x] \\ \text{rank}_p(x)=\beta}} \sum_{M \in (\Gamma(\delta_{i,j}(x))\mathcal{U}) \backslash \Gamma_n} \\ &\times \sum_{\lambda \in L_{i,j}} 1|_{k,\mathcal{M}}([\lambda, 0], 0_n], \begin{pmatrix} 1_n & p^{-1}x \\ 0 & 1_n \end{pmatrix} M)(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}). \end{aligned}$$

Now $\Gamma(\delta_{i,j}(x))\mathcal{U}$ is a subgroup of $\Gamma(\delta_{i,j})$. For any $\begin{pmatrix} A & B \\ 0_n & {}^tA^{-1} \end{pmatrix} \in \Gamma(\delta_{i,j})$ we have

$$\begin{aligned} & 1|_{k,\mathcal{M}}([\lambda, 0], 0_n], \begin{pmatrix} 1_n & p^{-1}x \\ 0_n & 1_n \end{pmatrix} \begin{pmatrix} A & B \\ 0_n & {}^tA^{-1} \end{pmatrix} M) \\ = & 1|_{k,\mathcal{M}}([\lambda, 0], 0_n], \begin{pmatrix} A & B \\ 0_n & {}^tA^{-1} \end{pmatrix} \begin{pmatrix} 1_n & p^{-1}A^{-1}x^tA^{-1} \\ 0_n & 1_n \end{pmatrix} M) \\ = & 1|_{k,\mathcal{M}}([({}^tA\lambda, {}^tB\lambda), 0_n], \begin{pmatrix} 1_n & p^{-1}A^{-1}x^tA^{-1} \\ 0_n & 1_n \end{pmatrix} M) \\ = & 1|_{k,\mathcal{M}}([({}^tA\lambda, 0), 0_n], \begin{pmatrix} 1_n & p^{-1}A^{-1}x^tA^{-1} \\ 0_n & 1_n \end{pmatrix} M), \end{aligned}$$

and ${}^tAL_{i,j} = L_{i,j}$. Moreover, when $\begin{pmatrix} A & B \\ 0_n & {}^tA^{-1} \end{pmatrix}$ runs over all elements in a complete set of representatives of $\Gamma(\delta_{i,j}(x))\mathcal{U} \backslash \Gamma(\delta_{i,j})$, then $A^{-1}x^tA^{-1}$ runs over all elements in the equivalent class $[x]$ (cf. [Ya 89, proof of Proposition 3.2]). Therefore, we have

$$\begin{aligned} & K_{i,j}^\beta(\tau, z) \\ = & p^{-k(2n-i-j+1)+(n-j)(n-i+1)} \sum_{\substack{x={}^tx \in (\mathbb{Z}/p\mathbb{Z})^{(n,n)} \\ x=\text{diag}(0_i, x', 0_{n-j}) \\ \text{rank}_p(x')=\beta}} \sum_{M \in \Gamma(\delta_{i,j}) \backslash \Gamma_n} \\ & \times \sum_{\lambda \in L_{i,j}} 1|_{k,\mathcal{M}}([\lambda, 0], 0_n], \begin{pmatrix} 1_n & p^{-1}x \\ 0 & 1_n \end{pmatrix} M)(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) \\ = & p^{-k(2n-i-j+1)+(n-j)(n-i+1)} \sum_{M \in \Gamma(\delta_{i,j}) \backslash \Gamma_n} \\ & \times \sum_{\lambda \in L_{i,j}} 1|_{k,\mathcal{M}}([\lambda, 0], 0_n], M)(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) \sum_{\substack{x={}^tx \in (\mathbb{Z}/p\mathbb{Z})^{(n,n)} \\ x=\text{diag}(0_i, x', 0_{n-j}) \\ \text{rank}_p(x')=\beta}} e\left(\frac{1}{p}\mathcal{M}^t\lambda x\lambda\right). \end{aligned}$$

□

5.2. The function $\tilde{K}_{i,j}^\beta$. The purpose of this subsection is to introduce functions $\tilde{K}_{i,j}^\beta$ and to express $E_{k,\mathcal{M}}^{(n)}|V_{\alpha,n-\alpha}(p^2)$ as a summation of $\tilde{K}_{i,j}^\beta$. Moreover, we shall show that $\tilde{K}_{i,j}^\beta$ is a summation of certain exponential functions with generalized Gauss sums (cf. Proposition 5.3).

We define

$$\begin{aligned} L_{i,j}^* &:= L_{i,j,\mathcal{M},p}^* \\ &= \left\{ \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} \in (p^{-1}\mathbb{Z})^{(n,2)} \left| \begin{array}{l} \lambda_1 \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1} \in \mathbb{Z}^{(i,2)}, \lambda_2 \in \mathbb{Z}^{(j-i,2)} \\ \lambda_3 \in (p^{-1}\mathbb{Z})^{(n-j,2)}, 2\lambda_2\mathcal{M}^t\lambda_3 \in \mathbb{Z}^{(j-i,n-j)} \\ \lambda_3\mathcal{M}^t\lambda_3 \in \mathbb{Z}^{(n-j,n-j)}, 2\lambda_3\mathcal{M} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{Z}^{(n-j,1)} \end{array} \right. \right\} \end{aligned}$$

and define a generalized Gauss sum

$$G_{\mathcal{M}}^{j-i,l}(\lambda_2) := \sum_{\substack{x'=tx' \in (\mathbb{Z}/p\mathbb{Z})^{(j-i,j-i)} \\ \text{rank}_p(x')=j-i-l}} e\left(\frac{1}{p}\mathcal{M}^t\lambda_2 x' \lambda_2\right)$$

for $\lambda_2 \in \mathbb{Z}^{(j-i,2)}$. We remark that a formula for this generalized Gauss sum is given by Saito [Sa 91]. We define

$$\tilde{K}_{i,j}^\beta(\tau, z) := \tilde{K}_{i,j,\mathcal{M},p}^\beta(\tau, z) = \sum_{u,v \in (\mathbb{Z}/p\mathbb{Z})^{(n,1)}} \left(K_{i,j}^\beta|_{k,\mathcal{M}[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}]} [((0,u), (0,v)), 0_n] \right) (\tau, z).$$

Proposition 5.3. *Let the notation be as above. Then*

$$E_{k,\mathcal{M}}^{(n)}|V_{\alpha,n-\alpha}(p^2) = \sum_{0 \leq i \leq j \leq n} \tilde{K}_{i,j}^{\alpha-i-n+j}(\tau, z),$$

where

$$\begin{aligned} \tilde{K}_{i,j}^{\alpha-i-n+j}(\tau, z) &= p^{-k(2n-i-j+1)+(n-j)(n-i+1)+2n-j} \\ &\quad \times \sum_{M \in \Gamma(\delta_{i,j}) \backslash \Gamma_n} \sum_{\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} \in L_{i,j}^*} \{1|_{k,\mathcal{M}}([(\lambda, 0), 0_n], M)\}(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) \\ &\quad \times \sum_{u_2 \in (\mathbb{Z}/p\mathbb{Z})^{(j-i,1)}} G_{\mathcal{M}}^{j-i,n-\alpha}(\lambda_2 + (0, u_2)). \end{aligned}$$

Proof. From the definition of $\tilde{K}_{i,j}^\beta$ and Lemma 5.2 we obtain

$$\begin{aligned} &\tilde{K}_{i,j}^{\alpha-i-n+j}(\tau, z) \\ &= p^{-k(2n-i-j+1)+(n-j)(n-i+1)} \sum_{M \in \Gamma(\delta_{i,j}) \backslash \Gamma_n} \sum_{\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} \in L_{i,j}} G_{\mathcal{M}}^{j-i,n-\alpha}(\lambda_2) \\ &\quad \times \sum_{u,v \in (\mathbb{Z}/p\mathbb{Z})^{(n,1)}} (1|_{k,\mathcal{M}}([(\lambda, 0), 0_n], M)(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}))|_{k,\mathcal{M}[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}]} [((0,u), (0,v)), 0_n], \end{aligned}$$

where, in the second summation, $\lambda_1 \in \mathbb{Z}^{(i,2)}$, $\lambda_2 \in \mathbb{Z}^{(j-i,2)}$, $\lambda_3 \in \mathbb{Z}^{(n-j,2)}$ and $\lambda \in L_{i,j}$.

By a straightforward calculation we have

$$1|_{k,\mathcal{M}}([(\lambda, 0), 0_n], M)(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) = 1|_{k,\mathcal{M}[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}]}([\lambda \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1}, 0), 0_n], M)(\tau, z).$$

Hence the last summation of the above identity is

$$\begin{aligned}
& \sum_{u,v \in (\mathbb{Z}/p\mathbb{Z})^{(n,1)}} \left\{ 1|_{k,\mathcal{M}}([(\lambda, 0), 0_n], M)(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix})|_{k,\mathcal{M}[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}]} [((0, u), (0, v)), 0_n] \right\} (\tau, z) \\
&= \sum_{u',v' \in (\mathbb{Z}/p\mathbb{Z})^{(n,1)}} \left\{ 1|_{k,\mathcal{M}[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}]} ([(\lambda \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1} + (0, u'), (0, v')), 0_n], M) \right\} (\tau, z) \\
&= \sum_{u',v' \in (\mathbb{Z}/p\mathbb{Z})^{(n,1)}} \left\{ 1|_{k,\mathcal{M}}([\lambda + (0, u'), (0, v')), 0_n], M) \right\} (\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) \\
&= \sum_{u' \in (\mathbb{Z}/p\mathbb{Z})^{(n,1)}} \left\{ 1|_{k,\mathcal{M}}([\lambda + (0, u'), 0), 0_n], M) \right\} (\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) \sum_{v' \in (\mathbb{Z}/p\mathbb{Z})^{(n,1)}} e(2\mathcal{M}^t \lambda(0, v')),
\end{aligned}$$

where, in the first identity, we used

$$(M, [((0, u), (0, v)), 0_n]) = ([((0, u'), (0, v')), 0_n], M)$$

with $\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$ for any $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$. Now, for $\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} \in L_{i,j}$ we have

$$\sum_{v' \in (\mathbb{Z}/p\mathbb{Z})^{(n,1)}} e(2\mathcal{M}^t \lambda(0, v')) = \begin{cases} p^n & \text{if } 2\lambda_3 \mathcal{M} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{Z}^{(n-j,1)}, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$\begin{aligned}
& \tilde{K}_{i,j}^{\alpha-i-n+j}(\tau, z) \\
&= p^{-k(2n-i-j+1)+(n-j)(n-i+1)+n} \sum_{M \in \Gamma(\delta_{i,j}) \setminus \Gamma_n} \sum_{\substack{\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} \in L_{i,j} \\ 2\lambda_3 \mathcal{M} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{Z}^{(n-j,1)}}} G_{\mathcal{M}}^{j-i, n-\alpha}(\lambda_2) \\
&\times \sum_{u \in (\mathbb{Z}/p\mathbb{Z})^{(n,1)}} \left\{ 1|_{k,\mathcal{M}}([\lambda + (0, u), 0), 0_n], M) \right\} (\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}).
\end{aligned}$$

Thus

$$\begin{aligned}
& \tilde{K}_{i,j}^{\alpha-i-n+j}(\tau, z) \\
&= p^{-k(2n-i-j+1)+(n-j)(n-i+1)+n} \sum_{M \in \Gamma(\delta_{i,j}) \setminus \Gamma_n} \sum_{\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} \in L_{i,j}^*} \\
&\times \left\{ 1|_{k,\mathcal{M}}([\lambda, 0], 0_n], M) \right\} (\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) \\
&\times p^{n-j} \sum_{u_2 \in (\mathbb{Z}/p\mathbb{Z})^{(j-i,1)}} G_{\mathcal{M}}^{j-i, n-\alpha}(\lambda_2 + (0, u_2)),
\end{aligned}$$

where $L_{i,j}^*$ is defined as before. □

6. PROOF OF THEOREM 1.1

The purpose of this section is to give the proof of Theorem 1.1. From Proposition 5.3 we recall

$$E_{k,\mathcal{M}}^{(1)}|V_{\alpha,1-\alpha}(p^2) = \sum_{0 \leq i \leq j \leq 1} \tilde{K}_{i,j}^{\alpha-i-1+j}.$$

Hence

$$E_{k,\mathcal{M}}^{(1)}|V_{1,0}(p^2) = \tilde{K}_{1,1}^0 + \tilde{K}_{0,1}^1 + \tilde{K}_{0,0}^0.$$

6.1. Calculation of $\tilde{K}_{i,j}^\beta$ for $n = 1$. In this subsection we calculate $\tilde{K}_{1,1}^0$, $\tilde{K}_{0,1}^1$ and $\tilde{K}_{0,0}^0$.

Lemma 6.1. *For $\mathcal{M} \in \text{Sym}_2^+$ let D_0 be the discriminant of $\mathbb{Q}(\sqrt{-\det(2\mathcal{M})})$. Let f be the positive integer such that $-\det(2\mathcal{M}) = D_0 f^2$. Then, for any prime p we obtain*

$$\begin{aligned} \tilde{K}_{1,1}^0(\tau, z) &= p^{-k+1} E_{k,\mathcal{M}[(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix})]}^{(1)}(\tau, z), \\ \tilde{K}_{0,1}^1(\tau, z) &= -p^{-2k+2} \left(\frac{D_0 f^2}{p} \right) E_{k,\mathcal{M}[(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix})]}^{(1)}(\tau, z) \\ &\quad + p^{-2k+2} \left(\frac{D_0 f^2}{p} \right) E_{k,\mathcal{M}}^{(1)}(\tau, z(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}))), \\ \tilde{K}_{0,0}^0(\tau, z) &= \begin{cases} p^{-3k+4} E_{k,\mathcal{M}[X^{-1}(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix})]^{-1}}^{(1)}(\tau, z(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}))^t X(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix})) & \text{if } p|f, \\ p^{-3k+4} E_{k,\mathcal{M}}^{(1)}(\tau, z(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}))) & \text{if } p \nmid f, \end{cases} \end{aligned}$$

where, in the case $p|f$, $X = (\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix})$ is a matrix such that $\mathcal{M}[X^{-1}(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix})]^{-1} \in \text{Sym}_2^+$.

Proof. By the definition of $\tilde{K}_{i,j}^{\alpha-i-n+j}$ we have

$$\begin{aligned} \tilde{K}_{1,1}^0 &= p^{-k+1} \sum_{M \in \Gamma(\delta_{1,1}) \backslash \Gamma_1} \sum_{\lambda_1 \in L_{1,1}^*} \{1|_{k,\mathcal{M}}(([\lambda_1, 0], 0], M)\}(\tau, z(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix})) \\ &= p^{-k+1} \sum_{M \in \Gamma_\infty^{(1)} \backslash \Gamma_1} \sum_{\lambda_1 \in p\mathbb{Z} \times \mathbb{Z}} \{1|_{k,\mathcal{M}}(([\lambda_1, 0], 0], M)\}(\tau, z(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix})) \\ &= p^{-k+1} \sum_{M \in \Gamma_\infty^{(1)} \backslash \Gamma_1} \sum_{\lambda \in \mathbb{Z}^{(1,2)}} \{1|_{k,\mathcal{M}}(([\lambda(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}), 0], 0], M)\}(\tau, z(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix})) \\ &= p^{-k+1} \sum_{M \in \Gamma_\infty^{(1)} \backslash \Gamma_1} \sum_{\lambda \in \mathbb{Z}^{(1,2)}} \left\{ 1|_{k,\mathcal{M}[(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix})]}([\lambda, 0], 0], M) \right\}(\tau, z) \\ &= p^{-k+1} E_{k,\mathcal{M}[(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix})]}^{(1)}(\tau, z). \end{aligned}$$

Now we shall calculate $\tilde{K}_{0,1}^1$. First, for any $\lambda \in \mathbb{Z}^{(1,2)}$ we have

$$\begin{aligned} G_{\mathcal{M}}^{1,0}(\lambda) &= \sum_{\substack{x \in \mathbb{Z}/p\mathbb{Z} \\ \text{rank}_p(x)=1}} e\left(\frac{1}{p}\mathcal{M}^t \lambda x \lambda\right) \\ &= \begin{cases} p-1 & \text{if } \lambda \mathcal{M}^t \lambda \equiv 0 \pmod{p}, \\ -1 & \text{if } \lambda \mathcal{M}^t \lambda \not\equiv 0 \pmod{p}. \end{cases} \end{aligned}$$

Hence we have

$$\sum_{u \in \mathbb{Z}/p\mathbb{Z}} G_{\mathcal{M}}^{1,0}(\lambda + (0, u)) = \begin{cases} 0 & \text{if } \lambda \in p\mathbb{Z} \times \mathbb{Z}, \\ \left(\frac{D_0 f^2}{p}\right) p & \text{if } \lambda \notin p\mathbb{Z} \times \mathbb{Z}. \end{cases}$$

Thus

$$\begin{aligned} \tilde{K}_{0,1}^1 &= p^{-2k+2} \sum_{M \in \Gamma(\delta_{0,1}) \setminus \Gamma_1} \sum_{\lambda_2 \in L_{0,1}^*} \{1|_{k,\mathcal{M}}([(\lambda_2, 0), 0], M)\}(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) \\ &\quad \times \sum_{u_2 \in \mathbb{Z}/p\mathbb{Z}} G_{\mathcal{M}}^{1,0}(\lambda_2 + (0, u_2)) \\ &= -p^{-2k+2} \left(\frac{D_0 f^2}{p}\right) \sum_{M \in \Gamma_{\infty}^{(1)} \setminus \Gamma_1} \sum_{\lambda_2 \in p\mathbb{Z} \times \mathbb{Z}} \{1|_{k,\mathcal{M}}([(\lambda_2, 0), 0], M)\}(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) \\ &\quad + p^{-2k+2} \left(\frac{D_0 f^2}{p}\right) \sum_{M \in \Gamma_{\infty}^{(1)} \setminus \Gamma_1} \sum_{\lambda_2 \in \mathbb{Z} \times \mathbb{Z}} \{1|_{k,\mathcal{M}}([(\lambda_2, 0), 0], M)\}(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}). \end{aligned}$$

Finally, we shall calculate $\tilde{K}_{0,0}^0$. We have

$$L_{0,0}^* = \left\{ \lambda_3 \in (p^{-1}\mathbb{Z})^{(1,2)} \mid \lambda_3 \mathcal{M}^t \lambda_3 \in \mathbb{Z}, \ 2\lambda_3 \mathcal{M} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{Z} \right\}.$$

We need to consider two cases: the case p is an odd prime and the case $p = 2$. When p is an odd prime, there exists a matrix $X = \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} \in \mathbb{Z}^{(2,2)}$ such that the matrix $\mathcal{M}$ has an expression $\mathcal{M} \equiv {}^t X \begin{pmatrix} 4^{-1} |D_0| f^2 & \\ & 1 \end{pmatrix} X \pmod{p}$. We have

$$L_{0,0}^* = \begin{cases} \left\{ \lambda_3 \in (p^{-1}\mathbb{Z})^{(1,2)} \mid \lambda_3 {}^t X \in \frac{1}{p}\mathbb{Z} \times \mathbb{Z} \right\} & \text{if } p \nmid f, \\ \mathbb{Z}^{(1,2)} & \text{if } p \nmid f. \end{cases}$$

Thus, if $p|f$, then

$$\begin{aligned}
\tilde{K}_{0,0}^0 &= p^{-3k+4} \sum_{M \in \Gamma(\delta_{0,0}) \setminus \Gamma_1} \sum_{\substack{\lambda_3 \\ \lambda_3^t X \in \frac{1}{p}\mathbb{Z} \times \mathbb{Z}}} \{1|_{k,\mathcal{M}}[(\lambda_3, 0), 0], M)\} (\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) \\
&= p^{-3k+4} \sum_{M \in \Gamma_\infty^{(1)} \setminus \Gamma_1} \sum_{\substack{\lambda_3 \\ \lambda_3^t X \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \in \mathbb{Z}^{(1,2)}}} \\
&\quad \times \left\{ 1|_{k,\mathcal{M}[X^{-1} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1}]}[(\lambda_3^t X \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, 0), 0], M \right\} (\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^t X \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) \\
&= p^{-3k+4} E_{k,\mathcal{M}[X^{-1} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1}] }^{(1)} (\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^t X \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}),
\end{aligned}$$

and, if $p \nmid f$, then

$$\begin{aligned}
\tilde{K}_{0,0}^0 &= p^{-3k+4} \sum_{M \in \Gamma(\delta_{0,0}) \setminus \Gamma_1} \sum_{\lambda_3 \in \mathbb{Z}^{(1,2)}} \{1|_{k,\mathcal{M}}[(\lambda_3, 0), 0], M)\} (\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) \\
&= p^{-3k+4} E_{k,\mathcal{M}}^{(1)} (\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}).
\end{aligned}$$

When $p = 2$, there exist a matrix $X = \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} \in \mathbb{Z}^{(2,2)}$ and an integer u such that the matrix $\mathcal{M}$ equals one of the following three forms:

$$\begin{aligned}
&{}^t X \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} X \text{ with } u \equiv 0, 1, 2 \pmod{4}, \\
&{}^t X \begin{pmatrix} u & 1 \\ 1 & 1 \end{pmatrix} X \text{ with } u \equiv 0 \pmod{4}, \text{ or} \\
&{}^t X \begin{pmatrix} u & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} X.
\end{aligned}$$

If $2|f$, then $\mathcal{M} = {}^t X \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} X$ with $u \equiv 0 \pmod{4}$ or $\mathcal{M} = {}^t X \begin{pmatrix} u & 1 \\ 1 & 1 \end{pmatrix} X$ with $u \equiv 0 \pmod{4}$. By a straightforward calculation we have

$$L_{0,0}^* = \begin{cases} \{ \lambda_3 \in (2^{-1}\mathbb{Z})^{(1,2)} \mid \lambda_3^t X \in \frac{1}{2}\mathbb{Z} \times \mathbb{Z} \} & \text{if } 2|f, \\ \mathbb{Z}^{(1,2)} & \text{if } 2 \nmid f, \end{cases}$$

where $X = \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}$ is a matrix such that $\mathcal{M} = {}^t X \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} X$ with $u \equiv 0 \pmod{4}$ or $= {}^t X \begin{pmatrix} u & 1 \\ 1 & 1 \end{pmatrix} X$ with $u \equiv 0 \pmod{4}$. Thus, if $2|f$, then

$$\begin{aligned}
\tilde{K}_{0,0}^0 &= 2^{-3k+4} \sum_{M \in \Gamma_\infty^{(1)} \setminus \Gamma_1} \sum_{\substack{\lambda_3 \\ \lambda_3^t X \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \in \mathbb{Z}^{(1,2)}}} \\
&\quad \times \left\{ 1|_{k,\mathcal{M}[X^{-1} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^{-1}]}[(\lambda_3^t X \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, 0), 0], M \right\} (\tau, z \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^t X \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}) \\
&= 2^{-3k+4} E_{k,\mathcal{M}[X^{-1} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^{-1}] }^{(1)} (\tau, z \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^t X \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}).
\end{aligned}$$

And, if $2 \nmid f$, then

$$\begin{aligned} \tilde{K}_{0,0}^0 &= 2^{-3k+4} \sum_{M \in \Gamma(\delta_{0,0}) \setminus \Gamma_1} \sum_{\lambda_3 \in \mathbb{Z}^{(1,2)}} \{1|_{k,\mathcal{M}}([(\lambda_3, 0), 0], M)\}(\tau, z \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}) \\ &= 2^{-3k+4} E_{k,\mathcal{M}}^{(1)}(\tau, z \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}). \end{aligned}$$

Hence we obtain the formula for $\tilde{K}_{0,0}^0$.

Therefore we conclude the lemma. $\square$

6.2. Proof of Theorem 1.1. In this subsection we conclude the proof of Theorem 1.1. We recall $\mathcal{M} = \begin{pmatrix} * & * \\ * & 1 \end{pmatrix} \in \text{Sym}_2^+$. We put $m = \det(2\mathcal{M})$.

We define $E_{k,m}^{(1)} := \iota_{\mathcal{M}}(E_{k,\mathcal{M}}^{(1)})$, where the map $\iota_{\mathcal{M}}$ is defined in §4.4. We remark that $E_{k,m}^{(1)}$ is well-defined, namely, if $\mathcal{N} = \begin{pmatrix} * & * \\ * & 1 \end{pmatrix} \in \text{Sym}_2^+$ and $\det(2\mathcal{N}) = m$, then $\iota_{\mathcal{N}}(E_{k,\mathcal{N}}^{(1)}) = E_{k,m}^{(1)}$. This fact follows from Proposition 4.3 and from the fact that there exists a matrix $X = \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}$ such that $\mathcal{N} = \mathcal{M}[X]$.

The form $e_{k,m}^{(1)} \in J_{k-\frac{1}{2},m}^{(1)*}$ was defined as the Fourier-Jacobi coefficient of generalized Cohen-Eisenstein series of degree 2 (cf. §1), and due to Lemma 4.2, we have $e_{k,m}^{(1)} = \iota_M(e_{k,\mathcal{M}}^{(1)})$. For the definition of $e_{k,\mathcal{M}}^{(1)}$, see §3.

Now, by Proposition 3.3 and Proposition 4.4, we have

$$\begin{aligned} e_{k,m}^{(1)}|V_p^{(1)} &= e_{k,m}^{(1)}|\tilde{V}_{1,0}(p^2) \\ &= p^{3k-4} \iota_{\mathcal{M}[\begin{pmatrix} p & \\ & 1 \end{pmatrix}]}(e_{k,\mathcal{M}}^{(1)}|V_{1,0}(p^2)) \\ &= p^{3k-4} \sum_{d|f} g_k\left(\frac{f}{d^2}\right) \iota_{\mathcal{M}[\begin{pmatrix} p & \\ & 1 \end{pmatrix}]} \left(E_{k,\mathcal{M}[{}^t W_d^{-1}]}^{(1)}(\tau, z W_d)|V_{1,0}(p^2) \right), \end{aligned}$$

where the symbols f and W_d are the same ones in Proposition 3.3. By the definition of index-shift maps we have

$$E_{k,\mathcal{M}[{}^t W_d^{-1}]}^{(1)}(\tau, z W_d)|V_{1,0}(p^2) = \left(E_{k,\mathcal{M}[{}^t W_d^{-1}]}^{(1)}|V_{1,0}(p^2) \right)(\tau, z W_d).$$

The form $E_{k,\mathcal{M}[{}^t W_d^{-1}]}^{(1)}|V_{1,0}(p^2)$ is a linear combination of Jacobi-Eisenstein series of matrix index (cf. Lemma 6.1.)

Due to Proposition 4.3 we have $\iota_{\mathcal{M}[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}]}(E_{k,\mathcal{M}}^{(n)}(*, * \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}))(\tau, z) = E_{k,m}^{(n)}(\tau, pz)$ and $\iota_{\mathcal{M}[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}]} \left(E_{k,\mathcal{M}[X^{-1} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1}]}^{(n)}(*, * \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} {}^t X \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) \right)(\tau, z) = E_{k,\frac{m}{p^2}}^{(n)}(\tau, p^2 z)$. By using these

identities and due to Lemma 6.1, we obtain

$$\begin{aligned}
& \iota_{\mathcal{M}[\iota W_d^{-1}(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix})]} \left(E_{k, \mathcal{M}[\iota W_d^{-1}]}^{(1)} | V_{1,0}(p^2) \right) \\
&= p^{-k+1} E_{k, \frac{mp^2}{d^2}}^{(1)}(\tau, dz) - p^{-2k+2} \left(\frac{D_0 f^2 / d^2}{p} \right) E_{k, \frac{mp^2}{d^2}}^{(1)}(\tau, dz) \\
&\quad + p^{-2k+2} \left(\frac{D_0 f^2 / d^2}{p} \right) E_{k, \frac{m}{d^2}}^{(1)}(\tau, pdz) \\
&\quad + \delta \left(p \middle| \frac{f}{d} \right) p^{-3k+4} E_{k, \frac{m}{p^2 d^2}}^{(1)}(\tau, p^2 dz) + \delta \left(p \nmid \frac{f}{d} \right) p^{-3k+4} E_{k, \frac{m}{d^2}}^{(1)}(\tau, pdz),
\end{aligned}$$

where $\delta(\mathcal{S}) = 1$ or 0 accordingly as the statement $\mathcal{S}$ is true or false, and where D_0 is the discriminant of $\mathbb{Q}(\sqrt{-m})$. Hence

$$\begin{aligned}
& e_{k,m}^{(1)} | V_p^{(1)} \\
&= p^{2k-3} \sum_{d|f} g_k \left(\frac{m}{d^2} \right) E_{k, \frac{mp^2}{d^2}}^{(1)}(\tau, dz) - p^{k-2} \sum_{d|f} g_k \left(\frac{m}{d^2} \right) \left(\frac{D_0 f^2 / d^2}{p} \right) E_{k, \frac{mp^2}{d^2}}^{(1)}(\tau, dz) \\
&\quad + p^{k-2} \sum_{d|f} g_k \left(\frac{m}{d^2} \right) \left(\frac{D_0 f^2 / d^2}{p} \right) E_{k, \frac{m}{d^2}}^{(1)}(\tau, pdz) \\
&\quad + \sum_{d|f} \delta \left(p \middle| \frac{f}{d} \right) g_k \left(\frac{m}{d^2} \right) E_{k, \frac{m}{p^2 d^2}}^{(1)}(\tau, p^2 dz) + \sum_{d|f} \delta \left(p \nmid \frac{f}{d} \right) g_k \left(\frac{m}{d^2} \right) E_{k, \frac{m}{d^2}}^{(1)}(\tau, pdz).
\end{aligned}$$

Because of Lemma 3.2 we obtain

$$\begin{aligned}
& e_{k,m}^{(1)} | V_p^{(1)} \\
&= \sum_{d|f} g_k \left(\frac{mp^2}{d^2} \right) E_{k, \frac{mp^2}{d^2}}^{(1)}(\tau, dz) + p^{k-2} \left(\frac{D_0}{p} \right) \sum_{\substack{d|f \\ \frac{f}{d} \not\equiv 0 \pmod{p}}} g_k \left(\frac{m}{d^2} \right) E_{k, \frac{m}{d^2}}^{(1)}(\tau, pdz) \\
&\quad + \delta(p|f) \sum_{\substack{d|f \\ \frac{f}{d} \equiv 0 \pmod{p}}} g_k \left(\frac{m}{d^2} \right) E_{k, \frac{m}{d^2 p^2}}^{(1)}(\tau, p^2 dz) + \sum_{\substack{d|f \\ \frac{f}{d} \not\equiv 0 \pmod{p}}} g_k \left(\frac{m}{d^2} \right) E_{k, \frac{m}{d^2}}^{(1)}(\tau, pdz).
\end{aligned}$$

By using Lemma 3.2 again, we have

$$\begin{aligned}
& e_{k,m}^{(1)}|V_p^{(1)} \\
&= \sum_{d|f} g_k\left(\frac{mp^2}{d^2}\right) E_{k,\frac{mp^2}{d^2}}^{(1)}(\tau, dz) + p^{k-2} \left(\frac{D_0}{p}\right) \sum_{\substack{d|f \\ \frac{f}{d} \not\equiv 0 \pmod{p}}} g_k\left(\frac{m}{d^2}\right) E_{k,\frac{m}{d^2}}^{(1)}(\tau, pdz) \\
&\quad + \delta(p|f) p^{2k-3} \sum_{d|\frac{f}{p}} g_k\left(\frac{m}{d^2 p^2}\right) E_{k,\frac{m}{d^2 p^2}}^{(1)}(\tau, p^2 dz) \\
&\quad - \delta(p|f) p^{k-2} \sum_{\substack{d>0 \\ pd|f}} \left(\frac{m/(dp)^2}{p}\right) g_k\left(\frac{m}{d^2 p^2}\right) E_{k,\frac{m}{d^2 p^2}}^{(1)}(\tau, p^2 dz) \\
&\quad + \sum_{\substack{d|f \\ \frac{f}{d} \not\equiv 0 \pmod{p}}} g_k\left(\frac{mp^2}{(pd)^2}\right) E_{k,\frac{mp^2}{(pd)^2}}^{(1)}(\tau, pdz) \\
&= \sum_{d|f} g_k\left(\frac{mp^2}{d^2}\right) E_{k,\frac{mp^2}{d^2}}^{(1)}(\tau, dz) + p^{k-2} \left(\frac{D_0}{p}\right) \sum_{\substack{d|f \\ \frac{f}{d} \not\equiv 0 \pmod{p}}} g_k\left(\frac{m}{d^2}\right) E_{k,\frac{m}{d^2}}^{(1)}(\tau, pdz) \\
&\quad + \delta(p|f) p^{2k-3} \sum_{d|\frac{f}{p}} g_k\left(\frac{m}{d^2 p^2}\right) E_{k,\frac{m}{d^2 p^2}}^{(1)}(\tau, p^2 dz) \\
&\quad - \delta(p|f) p^{k-2} \left(\frac{D_0}{p}\right) \sum_{\substack{d'|f \\ \frac{f}{d'} \not\equiv 0 \pmod{p}}} g_k\left(\frac{m}{d'^2}\right) E_{k,\frac{m}{d'^2}}^{(1)}(\tau, pd'z) \\
&\quad + \sum_{\substack{d'|pf \\ \frac{pf}{d'} \not\equiv 0 \pmod{p}}} g_k\left(\frac{mp^2}{d'^2}\right) E_{k,\frac{mp^2}{d'^2}}^{(1)}(\tau, d'z) \\
&= \sum_{d|fp} g_k\left(\frac{mp^2}{d^2}\right) E_{k,\frac{mp^2}{d^2}}^{(1)}(\tau, dz) + p^{k-2} \left(\frac{D_0 f^2}{p}\right) \sum_{d|f} g_k\left(\frac{m}{d^2}\right) E_{k,\frac{m}{d^2}}^{(1)}(\tau, pdz) \\
&\quad + \delta(p|f) p^{2k-3} \sum_{d|\frac{f}{p}} g_k\left(\frac{m}{d^2 p^2}\right) E_{k,\frac{m}{d^2 p^2}}^{(1)}(\tau, p^2 dz).
\end{aligned}$$

Because $e_{k,m}^{(1)}(\tau, z) = \sum_{d|f} g_k\left(\frac{m}{d^2}\right) E_{k,\frac{m}{d^2}}^{(1)}(\tau, dz)$, we conclude Theorem 1.1. $\square$

6.3. Proof of Corollary 1.3. In this subsection we shall show Corollary 1.3. Let $M_{k-\frac{1}{2}}^+(\Gamma_0(4))$ be the Kohnen plus-space of weight $k - \frac{1}{2}$. Let $g(\tau) = \sum_m c(m) e^{2\pi\sqrt{-1}m\tau}$ be the Fourier expansion of an element g in $M_{k-\frac{1}{2}}^+(\Gamma_0(4))$. For any prime p the Hecke operator $T_1(p^2)$ is defined by

$$\begin{aligned} & (g|T_1(p^2))(\tau) \\ &:= \sum_m \left(c(p^2m) + p^{k-2} \left(\frac{(-1)^{k-1}m}{p} \right) c(m) + p^{2k-3} c\left(\frac{m}{p^2}\right) \right) e^{2\pi\sqrt{-1}m\tau}. \end{aligned}$$

Hence, by the definition of $V_p^{(1)}$ and of $S_p^{(1)}$ and by substituting $z = 0$ to $e_{k,m}^{(1)}(\tau, z)$, we obtain

$$\begin{aligned} \left(e_{k,m}^{(1)}(*, 0)|T_1(p^2) \right)(\tau) &= \left(e_{k,m}^{(1)}|V_p^{(1)} \right)(\tau, 0) \\ &= \left(e_{k,m}^{(1)}|S_p^{(1)} \right)(\tau, 0) \\ &= e_{k,p^2m}^{(1)}(\tau, 0) + p^{k-2} \left(\frac{-m}{p} \right) e_{k,m}^{(1)}(\tau, 0) + p^{2k-3} e_{k,\frac{m}{p^2}}^{(1)}(\tau, 0). \end{aligned}$$

Therefore

$$\begin{aligned} & \mathcal{H}_{k-\frac{1}{2}}^{(2)} \left(\left(\begin{pmatrix} \tau & 0 \\ 0 & \omega \end{pmatrix} \right) \right) \Big|_{\tau} T_1(p^2) \\ &= \sum_m \left(\left(e_{k,m}^{(1)}(*, 0)|T_1(p^2) \right)(\tau) \right) e^{2\pi\sqrt{-1}m\omega} \\ &= \sum_m \left(e_{k,p^2m}^{(1)}(\tau, 0) + p^{k-2} \left(\frac{-m}{p} \right) e_{k,m}^{(1)}(\tau, 0) + p^{2k-3} e_{k,\frac{m}{p^2}}^{(1)}(\tau, 0) \right) e^{2\pi\sqrt{-1}m\omega} \\ &= \mathcal{H}_{k-\frac{1}{2}}^{(2)} \left(\left(\begin{pmatrix} \tau & 0 \\ 0 & \omega \end{pmatrix} \right) \right) \Big|_{\omega} T_1(p^2). \end{aligned}$$

□

7. PROOF OF THEOREM 1.4

In this section we shall give the proof of Theorem 1.4. We treat the case degree $n = 2$. For the sake of simplicity we abbreviate $E_{k,\mathcal{M}}^{(2)}$ (resp. $e_{k,\mathcal{M}}^{(2)}$) as $E_{k,\mathcal{M}}$ (resp. $e_{k,\mathcal{M}}$).

7.1. Calculation of $\tilde{K}_{i,j}^\beta$. In this subsection we shall express $\tilde{K}_{i,j}^\beta(\tau, z)$ (cf. Proposition 5.3) as a linear combination of three Jacobi-Eisenstein series $E_{k,\mathcal{M}[(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix})]}(\tau, z)$, $E_{k,\mathcal{M}}(\tau, z(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}))$ and $E_{k,\mathcal{M}[X^{-1}(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix})^{-1}]}(\tau, z(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix})^t X(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}))$, where $X = (\begin{smallmatrix} 1 & \\ x & 1 \end{smallmatrix})$ is a certain matrix depending on the choice of $\mathcal{M}$ and p .

From Proposition 5.3 we recall

$$E_{k,\mathcal{M}}|V_{\alpha,2-\alpha}(p^2) = \sum_{0 \leq i \leq j \leq 2} \tilde{K}_{i,j}^{\alpha-i-2+j}.$$

Hence,

$$E_{k,\mathcal{M}}|V_{1,1}(p^2) = \tilde{K}_{1,2}^0 + \tilde{K}_{0,1}^0 + \tilde{K}_{0,2}^1$$

and

$$E_{k,\mathcal{M}}|V_{2,0}(p^2) = \tilde{K}_{2,2}^0 + \tilde{K}_{1,1}^0 + \tilde{K}_{0,0}^0 + \tilde{K}_{1,2}^1 + \tilde{K}_{0,1}^1 + \tilde{K}_{0,2}^2.$$

Lemma 7.1. *For $\mathcal{M} \in \text{Sym}_2^+$ let D_0 be the discriminant of $\mathbb{Q}(\sqrt{-\det(2\mathcal{M})})$. Let f be the positive integer such that $-\det(2\mathcal{M}) = D_0 f^2$. Then, for any prime p we obtain*

$$\tilde{K}_{1,2}^0(\tau, z) = p^{-2k+3} E_{k,\mathcal{M}}(\tau, z \begin{pmatrix} p & \\ & 1 \end{pmatrix}) + p^{-2k+4} E_{k,\mathcal{M}[\begin{pmatrix} p & \\ & 1 \end{pmatrix}]}(\tau, z),$$

$$\begin{aligned} \tilde{K}_{0,1}^0(\tau, z) = & \\ & \begin{cases} p^{-4k+7} E_{k,\mathcal{M}[X^{-1} \begin{pmatrix} p & \\ & 1 \end{pmatrix}^{-1}]}(\tau, z \begin{pmatrix} p & \\ & 1 \end{pmatrix}^t X \begin{pmatrix} p & \\ & 1 \end{pmatrix}) + p^{-4k+8} E_{k,\mathcal{M}}(\tau, z \begin{pmatrix} p & \\ & 1 \end{pmatrix}) & \text{if } p|f, \\ p^{-4k+7} (p+1) E_{k,\mathcal{M}}(\tau, z \begin{pmatrix} p & \\ & 1 \end{pmatrix}) & \text{if } p \nmid f, \end{cases} \end{aligned}$$

where $X = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$ is a matrix such that $\mathcal{M}[X^{-1} \begin{pmatrix} p & \\ & 1 \end{pmatrix}^{-1}] \in \text{Sym}_2^+$,

$$\begin{aligned} & \tilde{K}_{0,2}^1(\tau, z) \\ = & -p^{-3k+5} \left(\frac{D_0 f^2}{p} \right) E_{k,\mathcal{M}[\begin{pmatrix} p & \\ & 1 \end{pmatrix}]}(\tau, z) + p^{-3k+5} \left(\frac{D_0 f^2}{p} \right) E_{k,\mathcal{M}}(\tau, z \begin{pmatrix} p & \\ & 1 \end{pmatrix}). \end{aligned}$$

Proof. Let $G_{\mathcal{M}}^{j-i,2-\alpha}(\lambda)$ and $L_{i,j}^*$ be the symbols defined in §5.2 and $\Gamma(\delta_{i,j})$ be the symbol defined in §5.1.

For $i = 1, j = 2$ we have

$$L_{1,2}^* = \left\{ \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \in \mathbb{Z}^{(2,2)} \mid \lambda_1 \in p\mathbb{Z} \times \mathbb{Z}, \lambda_2 \in \mathbb{Z} \times \mathbb{Z} \right\}.$$

Now we remark that the set $\left\{ \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}, \begin{pmatrix} & -1 \\ 1 & p \end{pmatrix} \mid x \pmod{p} \right\}$ is a complete set of representatives of $\delta_{1,2} GL_2(\mathbb{Z}) \delta_{1,2}^{-1} \cap GL_2(\mathbb{Z}) \backslash GL_2(\mathbb{Z})$. Hence, for any function F on $\mathbb{Z}^{(2,2)}$ we obtain

$$\begin{aligned} \sum_{\begin{pmatrix} A & B \\ 0_2 & tA^{-1} \end{pmatrix} \in \Gamma(\delta_{1,2}) \backslash \Gamma_{\infty}^{(2)}} \sum_{\lambda \in L_{1,2}^*} F(tA\lambda) &= \sum_{A \in \delta_{1,2} GL_2(\mathbb{Z}) \delta_{1,2}^{-1} \cap GL_2(\mathbb{Z}) \backslash GL_2(\mathbb{Z})} \sum_{\lambda \in L_{1,2}^*} F(tA\lambda) \\ &= \sum_{\lambda \in \mathbb{Z}^{(2,2)}} F(\lambda) + p \sum_{\lambda \in \mathbb{Z}^{(2,2)}} F(\lambda \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}), \end{aligned}$$

if the above summations are absolutely convergent (cf. Lemma 8.3 in the appendix). Due to Proposition 5.3 we therefore obtain

$$\begin{aligned} \tilde{K}_{1,2}^0(\tau, z) &= p^{-2k+2} \sum_{M \in \Gamma(\delta_{1,2}) \backslash \Gamma_2} \sum_{\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \in L_{1,2}^*} \{1|_{k, \mathcal{M}}([(\lambda, 0), 0_n], M)\}(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) \\ &\quad \times \sum_{u_2 \in (\mathbb{Z}/p\mathbb{Z})^{(1,1)}} G_{\mathcal{M}}^{1,1}(\lambda_2 + (0, u_2)). \end{aligned}$$

Because $G_{\mathcal{M}}^{1,1}(\lambda_2 + (0, u_2)) = 1$ for any $\lambda_2 \in \mathbb{Z}^{(1,2)}$, $u_2 \in \mathbb{Z}$, and because

$$\{1|_{k, \mathcal{M}}([(\lambda \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, 0), 0_2], M)\}(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) = \left\{1|_{k, \mathcal{M}[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}]}([(\lambda, 0), 0_2], M)\right\}(\tau, z),$$

we obtain

$$\begin{aligned} &\tilde{K}_{1,2}^0(\tau, z) \\ &= p^{-2k+3} \sum_{M \in \Gamma_{\infty}^{(2)} \backslash \Gamma_2} \sum_{M_0 \in \Gamma(\delta_{1,2}) \backslash \Gamma_{\infty}^{(2)}} \sum_{\lambda \in L_{1,2}^*} \{1|_{k, \mathcal{M}}([(\lambda, 0), 0_n], M_0 M)\}(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) \\ &= p^{-2k+3} \left\{ \sum_{M \in \Gamma_{\infty}^{(2)} \backslash \Gamma_2} \sum_{\lambda \in \mathbb{Z}^{(2,2)}} \{1|_{k, \mathcal{M}}([(\lambda, 0), 0_2], M)\}(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) \right. \\ &\quad \left. + p \sum_{M \in \Gamma_{\infty}^{(2)} \backslash \Gamma_2} \sum_{\lambda \in \mathbb{Z}^{(2,2)}} \{1|_{k, \mathcal{M}}([(\lambda \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, 0), 0_2], M)\}(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) \right\} \\ &= p^{-2k+3} E_{k, \mathcal{M}}(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) + p^{-2k+4} E_{k, \mathcal{M}[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}]}(\tau, z). \end{aligned}$$

Thus we have the formula for $\tilde{K}_{1,2}^0(\tau, z)$.

Now we shall calculate $\tilde{K}_{0,1}^0(\tau, z)$. If $p|f$, then we can take matrices $X = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \in \mathbb{Z}^{(2,2)}$ and $\mathcal{M}' \in \text{Sym}_2^+$ which satisfy $\mathcal{M} = \mathcal{M}'[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} X]$. Then

$$L_{0,1}^* = \left\{ \begin{pmatrix} \lambda_2 \\ \lambda_3 \end{pmatrix} \mid \lambda_2 \in \mathbb{Z}^{(1,2)}, \lambda_3 {}^t X \in \mathbb{Z}^{(1,2)} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1} \right\}$$

and

$$\begin{aligned} \tilde{K}_{0,1}^0(\tau, z) &= p^{-4k+6} \sum_{M \in \Gamma(\delta_{0,1}) \backslash \Gamma_2} \sum_{\lambda = \begin{pmatrix} \lambda_2 \\ \lambda_3 \end{pmatrix} \in L_{0,1}^*} \{1|_{k, \mathcal{M}}([(\lambda, 0), 0_2], M)\}(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) \\ &\quad \times \sum_{u_2 \in \mathbb{Z}/p\mathbb{Z}} G_{\mathcal{M}}^{1,1}(\lambda_2 + (0, u_2)) \\ &= p^{-4k+7} \sum_{M \in \Gamma(\delta_{0,1}) \backslash \Gamma_2} \sum_{\lambda \in L_{0,1}^*} \{1|_{k, \mathcal{M}}([(\lambda, 0), 0_2], M)\}(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}). \end{aligned}$$

Because

$$\begin{aligned} & \{1|_{k,\mathcal{M}}([\lambda, 0], 0_2], M)\}(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) \\ &= \{1|_{k,\mathcal{M}'}([\lambda^t X \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, 0], 0_2], M)\}(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^t X \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}), \end{aligned}$$

we obtain

$$\begin{aligned} & \tilde{K}_{0,1}^0(\tau, z) \\ &= p^{-4k+7} \sum_{M \in \Gamma(\delta_{0,1}) \setminus \Gamma_2} \sum_{\substack{\lambda \\ \lambda^t X \in \begin{pmatrix} p\mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}}} \{1|_{k,\mathcal{M}'}([\lambda^t X, 0], 0_2], M)\}(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^t X \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) \\ &= p^{-4k+7} \left\{ \sum_{M \in \Gamma_\infty^{(2)} \setminus \Gamma_2} \sum_{\substack{\lambda \\ \lambda^t X \in \mathbb{Z}^{(2,2)}}} \{1|_{k,\mathcal{M}'}([\lambda^t X, 0], 0_2], M)\}(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^t X \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) \right. \\ & \quad \left. + p \sum_{M \in \Gamma_\infty^{(2)} \setminus \Gamma_2} \sum_{\substack{\lambda \\ \lambda^t X \in \mathbb{Z}^{(2,2)}}} \{1|_{k,\mathcal{M}'}([\lambda^t X \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, 0], 0_2], M)\}(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^t X \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) \right\} \\ &= p^{-4k+7} E_{k,\mathcal{M}[X^{-1} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1}]}(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^t X \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) + p^{-4k+8} E_{k,\mathcal{M}}(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}). \end{aligned}$$

Thus we obtain the formula of $\tilde{K}_{0,1}^0(\tau, z)$ for the case $p|f$. If $p \nmid f$, then by a straightforward calculation, we have $L_{0,1}^* = \mathbb{Z}^{(2,2)}$. Hence

$$\begin{aligned} & \tilde{K}_{0,1}^0(\tau, z) \\ &= p^{-4k+6} \sum_{M \in \Gamma(\delta_{0,1}) \setminus \Gamma_2} \sum_{\lambda \in \mathbb{Z}^{(2,2)}} \{1|_{k,\mathcal{M}}([\lambda, 0], 0_2], M)\}(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) \\ & \quad \times \sum_{u_2 \in \mathbb{Z}/p\mathbb{Z}} G_{\mathcal{M}}^{1,1}(\lambda_2 + (0, u_2)) \\ &= p^{-4k+7} [\Gamma_\infty^{(2)} : \Gamma(\delta_{0,1})] \sum_{M \in \Gamma_\infty^{(2)} \setminus \Gamma_2} \sum_{\lambda \in \mathbb{Z}^{(2,2)}} \{1|_{k,\mathcal{M}}([\lambda, 0], 0_2], M)\}(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) \\ &= p^{-4k+7} (p+1) E_{k,\mathcal{M}}(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}). \end{aligned}$$

Here we used $[\Gamma_\infty^{(2)} : \Gamma(\delta_{0,1})] = [\mathrm{GL}_2(\mathbb{Z}) : \mathrm{GL}_2(\mathbb{Z}) \cap \delta_{0,1} \mathrm{GL}_2(\mathbb{Z}) \delta_{0,1}^{-1}] = p+1$. Thus we obtain the formula of $\tilde{K}_{0,1}^0(\tau, z)$ also for the case $p \nmid f$.

Finally we shall calculate $\tilde{K}_{0,2}^1(\tau, z)$. We remark that for a matrix $X = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$ and for a $\lambda \in \mathbb{Z}^{(2,2)}$, the condition $\lambda^t X \in \mathbb{Z}^{(2,2)} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ is equivalent to the condition $\lambda \in \mathbb{Z}^{(2,2)} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$.

Thus, due to Lemma 8.1 in the appendix, we have

$$\sum_{u_2 \in (\mathbb{Z}/p\mathbb{Z})^{(2,1)}} G_{\mathcal{M}}^{2,1}(\lambda + (0, u_2)) = \begin{cases} 0 & \text{if } \lambda \in \mathbb{Z}^{(2,2)} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \\ p^3 \left(\frac{D_0 f^2}{p} \right) & \text{if } \lambda \notin \mathbb{Z}^{(2,2)} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}. \end{cases}$$

Therefore

$$\begin{aligned} & \tilde{K}_{0,2}^1(\tau, z) \\ = & p^{-3k+2} \sum_{M \in \Gamma_{\infty}^{(2)} \setminus \Gamma_2} \sum_{\lambda \in \mathbb{Z}^{(2,2)}} \{1|_{k, \mathcal{M}}[(\lambda, 0), 0_2], M\}(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) \\ & \times \sum_{u_2 \in (\mathbb{Z}/p\mathbb{Z})^{(2,1)}} G_{\mathcal{M}}^{2,1}(\lambda + (0, u_2)) \\ = & p^{-3k+2} \sum_{M \in \Gamma_{\infty}^{(2)} \setminus \Gamma_2} \sum_{\substack{\lambda \in \mathbb{Z}^{(2,2)} \\ \lambda \notin \mathbb{Z}^{(2,2)} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}}} p^3 \left(\frac{D_0 f^2}{p} \right) \{1|_{k, \mathcal{M}}[(\lambda, 0), 0_2], M\}(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) \\ = & p^{-3k+5} \left(\frac{D_0 f^2}{p} \right) \left\{ - \sum_{M \in \Gamma_{\infty}^{(2)} \setminus \Gamma_2} \sum_{\lambda \in \mathbb{Z}^{(2,2)}} \{1|_{k, \mathcal{M}}[(\lambda \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, 0), 0_2], M\}(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) \right. \\ & \left. + \sum_{M \in \Gamma_{\infty}^{(2)} \setminus \Gamma_2} \sum_{\lambda \in \mathbb{Z}^{(2,2)}} \{1|_{k, \mathcal{M}}[(\lambda, 0), 0_2], M\}(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) \right\} \\ = & -p^{-3k+5} \left(\frac{D_0 f^2}{p} \right) E_{k, \mathcal{M}}[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}](\tau, z) + p^{-3k+5} \left(\frac{D_0 f^2}{p} \right) E_{k, \mathcal{M}}(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}). \end{aligned}$$

Hence we conclude the lemma. $\square$

Lemma 7.2. *For $\mathcal{M} \in \text{Sym}_2^+$ let D_0 and f be as in Lemma 7.1. For any prime p we obtain*

$$\begin{aligned}
\tilde{K}_{2,2}^0(\tau, z) &= p^{-k+2} E_{k, \mathcal{M}[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}]}(\tau, z), \\
\tilde{K}_{1,1}^0(\tau, z) &= \begin{cases} p^{-3k+6} E_{k, \mathcal{M}}(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) + p^{-3k+7} E_{k, \mathcal{M}[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}]}(\tau, z) & \text{if } p \nmid f, \\ p^{-3k+5} E_{k, \mathcal{M}'}(\tau, z \begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix}) + p^{-3k+5} (p-1) E_{k, \mathcal{M}}(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) \\ + p^{-3k+7} E_{k, \mathcal{M}[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}]}(\tau, z) & \text{if } p \mid f, \end{cases} \\
\tilde{K}_{0,0}^0(\tau, z) &= \begin{cases} p^{-5k+10} E_{k, \mathcal{M}}(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) & \text{if } p \nmid f, \\ p^{-5k+10} E_{k, \mathcal{M}'}(\tau, z \begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix}) & \text{if } p \mid f, \end{cases} \\
\tilde{K}_{1,2}^1(\tau, z) &= p^{-2k+3} \left(\frac{D_0 f^2}{p} \right) E_{k, \mathcal{M}}(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) \\ &\quad - p^{-2k+3} \left(\frac{D_0 f^2}{p} \right) E_{k, \mathcal{M}[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}]}(\tau, z), \\
\tilde{K}_{0,1}^1(\tau, z) &= p^{-4k+8} \left(\frac{D_0 f^2}{p} \right) E_{k, \mathcal{M}}(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) \\ &\quad - p^{-4k+8} \left(\frac{D_0 f^2}{p} \right) E_{k, \mathcal{M}[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}]}(\tau, z), \\
\tilde{K}_{0,2}^2(\tau, z) &= p^{-3k+4} (p-1) E_{k, \mathcal{M}}(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}).
\end{aligned}$$

Proof. For the calculation of $\tilde{K}_{i,j}^{j-i}$ we need to determine the set $L_{i,j}^*$, the value of the summation $\sum_{u_2 \in (\mathbb{Z}/p\mathbb{Z})^{(j-i,1)}} G_{\mathcal{M}}^{(j-i,0)}(\lambda + (0, u_2))$ and a complete set of the representatives of

$\Gamma(\delta_{i,j}) \backslash \Gamma_{\infty}^{(2)}$. The table of these are given in §8.3 in the appendix.

For the calculation of $\tilde{K}_{1,1}^0$ we use the identity

$$\begin{aligned}
&\sum_{\begin{pmatrix} A & B \\ 0_2 & tA^{-1} \end{pmatrix} \in \Gamma(\delta_{1,1}) \backslash \Gamma_{\infty}^{(2)}} \sum_{\lambda \in \begin{pmatrix} p^2\mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}} F(tA\lambda) \\
&= \sum_{\lambda \in \mathbb{Z}^{(2,2)}} F(\lambda) + (p-1) \sum_{\lambda \in \mathbb{Z}^{(2,2)}} F(\lambda \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) + p^2 \sum_{\lambda \in \mathbb{Z}^{(2,2)}} F(\lambda \begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix}),
\end{aligned}$$

where F is a function on $\mathbb{Z}^{(2,2)}$ such that the above summations are absolutely convergent. The proof of this identity will be given in Lemma 8.4 in the appendix.

The rest of the calculation is an analogue to Lemma 7.1, hence we omit the detail. $\square$

7.2. Proof of Theorem 1.4. In this subsection we conclude the proof of Theorem 1.4.

Let $\mathcal{M} = \begin{pmatrix} * & * \\ * & 1 \end{pmatrix} \in \text{Sym}_2^+$ be a matrix which satisfies $\det(2\mathcal{M}) = m$. Let D_0, f be as before, namely D_0 is the discriminant of $\mathbb{Q}(\sqrt{-m})$ and f is the positive integer which satisfies $-m = D_0 f^2$.

We define $E_{k,m} = E_{k,m}^{(2)} := \iota_{\mathcal{M}}(E_{k,\mathcal{M}}^{(2)})$, where the map $\iota_{\mathcal{M}}$ is defined in §4.4. We remark that $E_{k,m}$ is well-defined, namely it does not depend on the choice of $\mathcal{M}$ (cf. §6.2.)

The form $e_{k,m} := e_{k,m}^{(2)} \in J_{k-\frac{1}{2},m}^{(2)*}$ was defined as the Fourier-Jacobi coefficient of the generalized Cohen-Eisenstein series $\mathcal{H}_{k-\frac{1}{2}}^{(3)}$ of degree 3 (cf. §1). And, due to Lemma 4.2, we have $e_{k,m} = \iota_M(e_{k,\mathcal{M}})$.

By Proposition 4.4 and by Proposition 3.3, we have

$$\begin{aligned} & e_{k,m} | \tilde{V}_{\alpha,2-\alpha}(p^2) \\ &= p^{5k-11+\frac{1}{2}\alpha} \iota_{\mathcal{M}[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}]} (e_{k,\mathcal{M}} | V_{\alpha,2-\alpha}(p^2)) \\ &= p^{5k-11+\frac{1}{2}\alpha} \iota_{\mathcal{M}[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}]} \left(\sum_{d|f} g_k\left(\frac{m}{d^2}\right) \left(E_{k,\mathcal{M}[{}^t W_d^{-1}]} | V_{\alpha,2-\alpha}(p^2) \right) (\tau, z W_d) \right). \end{aligned}$$

The forms $E_{k,\mathcal{M}} | V_{\alpha,2-\alpha}(p^2)$ ($\alpha = 1, 2$) have been calculated in §7.1, and are written as linear combinations of three forms $E_{k,\mathcal{M}[X^{-1}(\begin{pmatrix} p & 1 \\ 1 & 1 \end{pmatrix})^{-1}]}(\tau, z(\begin{pmatrix} p & 1 \\ 1 & 1 \end{pmatrix})^t X(\begin{pmatrix} p & 1 \\ 1 & 1 \end{pmatrix}))$, $E_{k,\mathcal{M}}(\tau, z(\begin{pmatrix} p & 1 \\ 1 & 1 \end{pmatrix}))$ and $E_{k,\mathcal{M}[\begin{pmatrix} p & 1 \\ 1 & 1 \end{pmatrix}]}(\tau, z)$. We recall the definitions of $V_{1,p}^{(2)}$ and $V_{2,p}^{(2)}$:

$$\begin{aligned} e_{k,m} | V_{1,p}^{(2)} &= p^{-k+\frac{7}{2}} e_{k,m} | \tilde{V}_{1,1}(p^2), \\ e_{k,m} | V_{2,p}^{(2)} &= e_{k,m} | \tilde{V}_{2,0}(p^2). \end{aligned}$$

Because we defined $E_{k,m} = \iota_{\mathcal{M}}(E_{k,\mathcal{M}})$, we have

$$\begin{aligned} E_{k,\frac{mp^2}{d^2}}(\tau, dz) &= \iota_{\mathcal{M}[\begin{pmatrix} p & 1 \\ 1 & 1 \end{pmatrix}]} (E_{k,\mathcal{M}[{}^t W_d^{-1}(\begin{pmatrix} p & 1 \\ 1 & 1 \end{pmatrix})]}(*, *W_d)(\tau, z), \\ E_{k,\frac{m}{d^2}}(\tau, pdz) &= \iota_{\mathcal{M}[\begin{pmatrix} p & 1 \\ 1 & 1 \end{pmatrix}]} (E_{k,\mathcal{M}[{}^t W_d^{-1}]}(*, *W_d(\begin{pmatrix} p & 1 \\ 1 & 1 \end{pmatrix}))) (\tau, z) \end{aligned}$$

and

$$E_{k,\frac{m}{p^2 d^2}}(\tau, p^2 dz) = \iota_{\mathcal{M}[\begin{pmatrix} p & 1 \\ 1 & 1 \end{pmatrix}]} (E_{k,\mathcal{M}[{}^t W_d^{-1} X^{-1}(\begin{pmatrix} p & 1 \\ 1 & 1 \end{pmatrix})^{-1}]}(*, *W_d(\begin{pmatrix} p & 1 \\ 1 & 1 \end{pmatrix})^t X(\begin{pmatrix} p & 1 \\ 1 & 1 \end{pmatrix}))) (\tau, z).$$

Now we shall calculate $e_{k,m} | V_{1,p}^{(2)}$. Due to Lemma 7.1 and due to the above identities, we have

$$e_{k,m} | V_{1,p}^{(2)} = A_1 + A_2 + A_3,$$

where

$$\begin{aligned}
A_1 &:= \sum_{d|f} \{p^{2k-4} E_{k, \frac{m}{d^2}}(\tau, pdz) + p^{2k-3} E_{k, \frac{mp^2}{d^2}}(\tau, dz)\} g_k\left(\frac{m}{d^2}\right), \\
A_2 &:= \sum_{\substack{d|f \\ \frac{f}{d} \equiv 0 \pmod{p}}} \{E_{k, \frac{m}{p^2 d^2}}(\tau, p^2 dz) + p E_{k, \frac{m}{d^2}}(\tau, pdz)\} g_k\left(\frac{m}{d^2}\right) \\
&\quad + \sum_{\substack{d|f \\ \frac{f}{d} \not\equiv 0 \pmod{p}}} (p+1) E_{k, \frac{m}{d^2}}(\tau, pdz) g_k\left(\frac{m}{d^2}\right)
\end{aligned}$$

and

$$\begin{aligned}
A_3 &:= \sum_{d|f} \left\{ -p^{k-2} \left(\frac{D_0 f^2 / d^2}{p} \right) E_{k, \frac{mp^2}{d^2}}(\tau, dz) \right. \\
&\quad \left. + p^{k-2} \left(\frac{D_0 f^2 / d^2}{p} \right) E_{k, \frac{m}{d^2}}(\tau, pdz) \right\} g_k\left(\frac{m}{d^2}\right).
\end{aligned}$$

By using Lemma 3.2 we have

$$\begin{aligned}
A_1 &= p^{2k-4} \sum_{d|f} E_{k, \frac{m}{d^2}}(\tau, pdz) g_k\left(\frac{m}{d^2}\right) + \sum_{d|f} E_{k, \frac{mp^2}{d^2}}(\tau, dz) g_k\left(\frac{mp^2}{d^2}\right) \\
&\quad + \left(\frac{D_0}{p} \right) p^{k-2} \sum_{\substack{d|f \\ \frac{f}{d} \not\equiv 0 \pmod{p}}} E_{k, \frac{mp^2}{d^2}}(\tau, dz) g_k\left(\frac{m}{d^2}\right),
\end{aligned}$$

$$\begin{aligned}
A_2 &= \delta(p|f)p^{2k-3} \sum_{d|\frac{f}{p}} E_{k, \frac{m}{p^2 d^2}}(\tau, p^2 dz) g_k\left(\frac{m}{d^2 p^2}\right) \\
&\quad - \delta(p|f) \sum_{d|\frac{f}{p}} p^{k-2} \left(\frac{D_0 f^2 / d^2 p^2}{p}\right) E_{k, \frac{m}{p^2 d^2}}(\tau, p^2 dz) g_k\left(\frac{m}{d^2 p^2}\right) \\
&\quad + \delta(p|f)p \sum_{\substack{d|f \\ \frac{f}{d} \equiv 0 \pmod{p}}} E_{k, \frac{m}{d^2}}(\tau, pdz) g_k\left(\frac{m}{d^2}\right) \\
&\quad + p \sum_{\substack{d|f \\ \frac{f}{d} \not\equiv 0 \pmod{p}}} E_{k, \frac{m}{d^2}}(\tau, pdz) g_k\left(\frac{m}{d^2}\right) + \sum_{\substack{d|f \\ \frac{f}{d} \not\equiv 0 \pmod{p}}} E_{k, \frac{m}{d^2}}(\tau, pdz) g_k\left(\frac{m}{d^2}\right) \\
&= \delta(p|f)p^{2k-3} \sum_{d|\frac{f}{p}} E_{k, \frac{m}{p^2 d^2}}(\tau, p^2 dz) g_k\left(\frac{m}{d^2 p^2}\right) \\
&\quad - \delta(p|f) \left(\frac{D_0}{p}\right) p^{k-2} \sum_{\substack{d|\frac{f}{p} \\ \frac{f}{dp} \not\equiv 0 \pmod{p}}} E_{k, \frac{m}{p^2 d^2}}(\tau, p^2 dz) g_k\left(\frac{m}{d^2 p^2}\right) \\
&\quad + p \sum_{d|f} E_{k, \frac{m}{d^2}}(\tau, pdz) g_k\left(\frac{m}{d^2}\right) + \sum_{\substack{d|pf \\ \frac{pf}{d} \not\equiv 0 \pmod{p}}} E_{k, \frac{p^2 m}{d^2}}(\tau, dz) g_k\left(\frac{p^2 m}{d^2}\right),
\end{aligned}$$

where we used the identities

$$\left(\frac{D_0 f^2 / d^2 p^2}{p}\right) = \left(\frac{D_0}{p}\right) \delta\left(p \nmid \frac{f}{pd}\right)$$

and

$$\delta(p|f)p \sum_{\substack{d|f \\ \frac{f}{d} \equiv 0 \pmod{p}}} + p \sum_{\substack{d|f \\ \frac{f}{d} \not\equiv 0 \pmod{p}}} = p \sum_{d|f},$$

and we have

$$\begin{aligned}
A_3 &= -\left(\frac{D_0}{p}\right) p^{k-2} \sum_{\substack{d|f \\ \frac{f}{d} \not\equiv 0 \pmod{p}}} E_{k, \frac{mp^2}{d^2}}(\tau, dz) g_k\left(\frac{m}{d^2}\right) \\
&\quad + \left(\frac{D_0}{p}\right) p^{k-2} \sum_{\substack{d|f \\ \frac{f}{d} \not\equiv 0 \pmod{p}}} E_{k, \frac{m}{d^2}}(\tau, pdz) g_k\left(\frac{m}{d^2}\right).
\end{aligned}$$

Thus, due to Proposition 3.3, we obtain

$$\begin{aligned}
e_{k,m}|V_{1,p}^{(2)} &= p^{2k-4} \sum_{d|f} E_{k,\frac{m}{d^2}}(\tau, pdz) g_k\left(\frac{m}{d^2}\right) + p \sum_{d|f} E_{k,\frac{m}{d^2}}(\tau, pdz) g_k\left(\frac{m}{d^2}\right) \\
&\quad + \sum_{d|f} E_{k,\frac{mp^2}{d^2}}(\tau, dz) g_k\left(\frac{mp^2}{d^2}\right) + \sum_{\substack{d|pf \\ \frac{pf}{d} \not\equiv 0 \pmod p}} E_{k,\frac{p^2m}{d^2}}(\tau, dz) g_k\left(\frac{p^2m}{d^2}\right) \\
&\quad + \delta(p|f) p^{2k-3} \sum_{d|\frac{f}{p}} E_{k,\frac{m}{p^2d^2}}(\tau, p^2dz) g_k\left(\frac{m}{d^2p^2}\right) \\
&\quad - \delta(p|f) \left(\frac{D_0}{p}\right) p^{k-2} \sum_{\substack{d|\frac{f}{p} \\ \frac{f}{dp} \not\equiv 0 \pmod p}} E_{k,\frac{m}{p^2d^2}}(\tau, p^2dz) g_k\left(\frac{m}{d^2p^2}\right) \\
&\quad + \left(\frac{D_0}{p}\right) p^{k-2} \sum_{\substack{d|f \\ \frac{f}{d} \not\equiv 0 \pmod p}} E_{k,\frac{m}{d^2}}(\tau, pdz) g_k\left(\frac{m}{d^2}\right) \\
&= p^{2k-4} \sum_{d|f} E_{k,\frac{m}{d^2}}(\tau, pdz) g_k\left(\frac{m}{d^2}\right) + p \sum_{d|f} E_{k,\frac{m}{d^2}}(\tau, pdz) g_k\left(\frac{m}{d^2}\right) \\
&\quad + \sum_{d|fp} E_{k,\frac{mp^2}{d^2}}(\tau, dz) g_k\left(\frac{mp^2}{d^2}\right) \\
&\quad + \delta(p|f) p^{2k-3} \sum_{d|\frac{f}{p}} E_{k,\frac{m}{p^2d^2}}(\tau, p^2dz) g_k\left(\frac{m}{d^2p^2}\right) \\
&\quad + \left(\frac{D_0f^2}{p}\right) p^{k-2} \sum_{d|f} E_{k,\frac{m}{d^2}}(\tau, pdz) g_k\left(\frac{m}{d^2}\right) \\
&= p(p^{2k-5} + 1) e_{k,m}(\tau, pz) \\
&\quad + e_{k,mp^2}(\tau, z) + p^{2k-3} e_{k,\frac{m}{p^2}}(\tau, p^2z) + \left(\frac{-m}{p}\right) p^{k-2} e_{k,m}(\tau, pz).
\end{aligned}$$

Hence we obtain the identity for $e_{k,m}|V_{1,p}^{(2)}$.

Because the calculation of $e_{k,m}|V_{2,p}^{(2)}$ is an analogue to the case of $e_{k,m}|V_{1,p}^{(2)}$, we omit the detail. $\square$

7.3. Proof of Corollary 1.5. In this subsection we shall show Corollary 1.5.

By the definition of $V_{1,p}^{(2)}$, $T_{2,1}(p^2)$ and of $S_p^{(2)}$ and by substituting $z = 0$ to $e_{k,m}^{(2)}(\tau, z)$, we obtain

$$\begin{aligned}
\left(e_{k,m}^{(2)}(*, 0) | T_{2,1}(p^2) \right) (\tau) &= \left(e_{k,m}^{(2)} | V_{1,p}^{(2)} \right) (\tau, 0) \\
&= \left(e_{k,m}^{(2)} | \left(p(p^{2k-5} + 1) + S_p^{(2)} \right) \right) (\tau, 0) \\
&= p(p^{2k-5} + 1) e_{k,m}^{(2)}(\tau, 0) + e_{k,p^2m}^{(2)}(\tau, 0) \\
&\quad + p^{k-2} \left(\frac{-m}{p} \right) e_{k,m}^{(2)}(\tau, 0) + p^{2k-3} e_{k,\frac{m}{p^2}}^{(2)}(\tau, 0).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\mathcal{H}_{k-\frac{1}{2}}^{(3)} \left(\begin{pmatrix} \tau & 0 \\ 0 & \omega \end{pmatrix} \right) \Big|_{\tau} T_{2,1}(p^2) \\
&= \sum_m \left(\left(e_{k,m}^{(2)}(*, 0) | T_{2,1}(p^2) \right) (\tau) \right) e^{2\pi\sqrt{-1}m\omega} \\
&= \sum_m \left(p(p^{2k-5} + 1) e_{k,m}^{(2)}(\tau, 0) \right. \\
&\quad \left. + e_{k,p^2m}^{(2)}(\tau, 0) + p^{k-2} \left(\frac{-m}{p} \right) e_{k,m}^{(2)}(\tau, 0) + p^{2k-3} e_{k,\frac{m}{p^2}}^{(2)}(\tau, 0) \right) e^{2\pi\sqrt{-1}m\omega} \\
&= \mathcal{H}_{k-\frac{1}{2}}^{(3)} \left(\begin{pmatrix} \tau & 0 \\ 0 & \omega \end{pmatrix} \right) \Big|_{\omega} (p(p^{2k-5} + 1) + T_1(p^2)).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\mathcal{H}_{k-\frac{1}{2}}^{(3)} \left(\begin{pmatrix} \tau & 0 \\ 0 & \omega \end{pmatrix} \right) \Big|_{\tau} T_{2,2}(p^2) \\
&= \mathcal{H}_{k-\frac{1}{2}}^{(3)} \left(\begin{pmatrix} \tau & 0 \\ 0 & \omega \end{pmatrix} \right) \Big|_{\omega} ((p^{2k-4} - p^{2k-6}) + p(p^{2k-5} + 1)T_1(p^2)).
\end{aligned}$$

□

8. APPENDIX

8.1. Values of some generalized Gauss sums. In this subsection we shall give the values of generalized Gauss sums $G_{\mathcal{M}}^{2,1}(\lambda)$ and $G_{\mathcal{M}}^{2,0}(\lambda)$, which are defined in §5.2. For odd primes these values follow from the result in [Sa 91]. We need these values for the calculation of $\tilde{K}_{i,j}^{\beta}$ in §7.1.

In this subsection we fix a matrix $\mathcal{M} = \begin{pmatrix} * & * \\ * & 1 \end{pmatrix} \in \text{Sym}_2^*$.

Lemma 8.1. *Let p be an odd prime and $X = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \in \mathbb{Z}^{(2,2)}$ be a matrix such that $\mathcal{M} \equiv {}^tX \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} X \pmod{p}$. Then, for $\lambda \in \mathbb{Z}^{(2,2)}$ we have*

$$G_{\mathcal{M}}^{2,1}(\lambda) = \sum_{\substack{x \in \text{Sym}_2(\mathbb{Z}/p\mathbb{Z}) \\ \text{rank}_p(x)=1}} e\left(\frac{1}{p}\mathcal{M}^t\lambda x\lambda\right) \\ = \begin{cases} p^2 - 1 & \text{if } \text{rank}_p(\lambda^t X) = 0, \\ p^2 - 1 & \text{if } \text{rank}_p(\lambda^t X) = 1 \text{ and } \lambda^t X \equiv (\lambda', t\lambda') \pmod{p} \\ & \text{with } t \text{ such that } u + t^2 \in p\mathbb{Z}, \\ -1 & \text{if } \text{rank}_p(\lambda^t X) = 1 \text{ and } \lambda^t X \equiv (\lambda', t\lambda') \pmod{p} \\ & \text{with } t \text{ such that } u + t^2 \notin p\mathbb{Z}, \\ -1 & \text{if } \lambda^t X \equiv (0, \lambda') \pmod{p} \text{ with } \lambda' \notin (p\mathbb{Z})^{(2,1)}, \\ \left(\frac{-u}{p}\right)p - 1 & \text{if } \text{rank}_p(\lambda^t X) = 2. \end{cases}$$

For $p = 2$ there exists a matrix $X = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$ such that $\mathcal{M} = {}^tX \begin{pmatrix} u & \frac{r}{2} \\ \frac{r}{2} & 1 \end{pmatrix} X$ with $r = 0$ or 1 . Then, for $\lambda \in \mathbb{Z}^{(2,2)}$ we have

$$G_{\mathcal{M}}^{2,1}(\lambda) = \sum_{\substack{x \in \text{Sym}_2(\mathbb{Z}/2\mathbb{Z}) \\ \text{rank}_2(x)=1}} e\left(\frac{1}{2}\mathcal{M}^t\lambda x\lambda\right) \\ = \begin{cases} 3 & \text{if } \text{rank}_2(\lambda^t X) = 0, \\ -1 + 2(1 + (-1)^{u+t}) & \text{if } \text{rank}_2(\lambda^t X) = 1 \\ & \text{and } \lambda^t X \equiv (\lambda', t\lambda') \pmod{2} \\ & \text{and } {}^tX^{-1}\mathcal{M}X^{-1} = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}, \\ -1 + 2(1 + (-1)^u) & \text{if } \text{rank}_2(\lambda^t X) = 1 \\ & \text{and } \lambda^t X \equiv (\lambda', t\lambda') \pmod{2} \\ & \text{and } {}^tX^{-1}\mathcal{M}X^{-1} = \begin{pmatrix} u & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}, \\ -1 & \text{if } \text{rank}_2(\lambda^t X) = 1 \\ & \text{and } \lambda^t X \equiv (0, \lambda') \pmod{2}, \\ -1 & \text{if } \text{rank}_2(\lambda^t X) = 2 \\ & \text{and } {}^tX^{-1}\mathcal{M}X^{-1} = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}, \\ 1 - 2(1 - (-1)^u) & \text{if } \text{rank}_2(\lambda^t X) = 2 \\ & \text{and } {}^tX^{-1}\mathcal{M}X^{-1} = \begin{pmatrix} u & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}. \end{cases}$$

Proof. For odd prime p this lemma follows from [Sa 91, Proposition 1.12].

For the case $p = 2$ we can directly calculate $G_{\mathcal{M}}^{2,1}(\lambda)$. The details is omitted here. $\square$

Lemma 8.2. *Let p be an odd prime and $X = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \in \mathbb{Z}^{(2,2)}$ be a matrix such that $\mathcal{M} \equiv {}^tX \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} X \pmod{p}$. Then, for $\lambda \in \mathbb{Z}^{(2,2)}$ we have*

$$\begin{aligned}
& G_{\mathcal{M}}^{2,0}(\lambda) \\
&= \sum_{\substack{x \in \text{Sym}_2(\mathbb{Z}/p\mathbb{Z}) \\ \text{rank}_p(x)=2}} e\left(\frac{1}{p} \mathcal{M}^t \lambda x \lambda\right) \\
&= \begin{cases} p^2(p-1) & \text{if } \text{rank}_p(\lambda^t X) = 0, \\ p^2(p-1) & \text{if } \text{rank}_2(\lambda^t X) = 1 \\ & \text{and } \lambda^t X \equiv (\lambda', t\lambda') \pmod{p} \text{ with } t \text{ such that } u+t^2 \in p\mathbb{Z}, \\ 0 & \text{if } \text{rank}_2(\lambda^t X) = 1 \\ & \text{and } \lambda^t X \equiv (\lambda', t\lambda') \pmod{p} \text{ with } t \text{ such that } u+t^2 \notin p\mathbb{Z}, \\ 0 & \text{if } \lambda^t X \equiv (0, \lambda') \pmod{p} \text{ with } \lambda' \notin (p\mathbb{Z})^{(2,1)}, \\ -\left(\frac{-u}{p}\right)p & \text{if } \text{rank}_p(\lambda^t X) = 2. \end{cases}
\end{aligned}$$

For $p = 2$ there exists a matrix $X = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$ such that $\mathcal{M} = {}^tX \begin{pmatrix} u & \frac{r}{2} \\ \frac{r}{2} & 1 \end{pmatrix} X$ with $r = 0$ or 1 . Then, for $\lambda \in \mathbb{Z}^{(2,2)}$ we have

$$\begin{aligned}
G_{\mathcal{M}}^{2,0}(\lambda) &= \sum_{\substack{x \in \text{Sym}_2(\mathbb{Z}/2\mathbb{Z}) \\ \text{rank}_2(x)=1}} e\left(\frac{1}{2} \mathcal{M}^t \lambda x \lambda\right) \\
&= \begin{cases} 4 & \text{if } \text{rank}_2(\lambda^t X) = 0, \\ 2(1 + (-1)^{u+t}) & \text{if } \text{rank}_2(\lambda^t X) = 1 \\ & \text{and } \lambda^t X \equiv (\lambda', t\lambda') \pmod{2} \\ & \text{and } {}^tX^{-1} \mathcal{M} X^{-1} = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}, \\ 2(1 + (-1)^u) & \text{if } \text{rank}_2(\lambda^t X) = 1 \\ & \text{and } \lambda^t X \equiv (\lambda', t\lambda') \pmod{2} \\ & \text{and } {}^tX^{-1} \mathcal{M} X^{-1} = \begin{pmatrix} u & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}, \\ 0 & \text{if } \text{rank}_2(\lambda^t X) = 1 \\ & \text{and } \lambda^t X \equiv (0, \lambda') \pmod{2}, \\ 0 & \text{if } \text{rank}_2(\lambda^t X) = 2 \\ & \text{and } {}^tX^{-1} \mathcal{M} X^{-1} = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}, \\ -2(-1)^u & \text{if } \text{rank}_2(\lambda^t X) = 2 \\ & \text{and } {}^tX^{-1} \mathcal{M} X^{-1} = \begin{pmatrix} u & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}. \end{cases}
\end{aligned}$$

Proof. For odd prime p , this lemma follows from [Sa 91, Theorem 1.3]. For the case $p = 2$ we can directly calculate $G_{\mathcal{M}}^{2,0}(\lambda)$. We omit the detail. $\square$

8.2. Index-shift maps $V_{1,p}^{(2)}$, $V_{2,p}^{(2)}$ and Fourier coefficients. Let

$$\phi(\tau, z) = \sum_{\substack{T \in \text{Sym}_2^*, S \in \mathbb{Z}^{(2,1)} \\ 4Tm - S^t S \geq 0}} C(T, S) e(T\tau + S^t z)$$

be the Fourier expansion of $\phi \in J_{k-\frac{1}{2}, m}^{(2)*}$. In this subsection we shall express Fourier coefficients of $\phi|V_{1,p}^{(2)}$ and $\phi|V_{2,p}^{(2)}$ as a linear combination of $C(T, S)$.

For any prime p we put

$$\begin{aligned} R(p) &:= \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mid x \bmod p \right\}, \\ R(p^2) &:= \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} py & 1 \\ -1 & 0 \end{pmatrix} \mid x \bmod p^2, y \bmod p \right\}. \end{aligned}$$

Then, the action of the index-shift maps $V_{1,p}^{(2)}$ and $V_{2,p}^{(2)}$ can be written as

$$\begin{aligned} (\phi|V_{1,p}^{(2)})(\tau, z) &= \sum_{\substack{T \in \text{Sym}_2^*, S \in \mathbb{Z}^{(2,1)} \\ 4Tmp^2 - S^t S \geq 0}} \sum_{i,j} \alpha_{1,i,j}(T, S) e(T\tau + Sz), \\ (\phi|V_{2,p}^{(2)})(\tau, z) &= \sum_{\substack{T \in \text{Sym}_2^*, S \in \mathbb{Z}^{(2,1)} \\ 4Tmp^2 - S^t S \geq 0}} \sum_{i,j} \alpha_{2,i,j}(T, S) e(T\tau + Sz), \end{aligned}$$

where, for odd prime p , we have

$$\begin{aligned} \alpha_{1,1,0}(T, S) &= p^{2k-4} \sum_{U \in R(p)} C \left(\begin{pmatrix} p^{-1} & \\ & 1 \end{pmatrix} U T^t U \begin{pmatrix} p^{-1} & \\ & 1 \end{pmatrix}, \frac{1}{p} \begin{pmatrix} p^{-1} & \\ & 1 \end{pmatrix} U S \right), \\ \alpha_{1,1,1}(T, S) &= \sum_{U \in R(p)} C \left(\begin{pmatrix} 1 & \\ & p \end{pmatrix} U T^t U \begin{pmatrix} 1 & \\ & p \end{pmatrix}, \frac{1}{p} \begin{pmatrix} 1 & \\ & p \end{pmatrix} U S \right), \\ \alpha_{1,2,0}(T, S) &= \begin{cases} \left(\frac{-a}{p} \right) p^{k-2} C(T, \frac{1}{p} S) & \text{if } p \nmid a \text{ and } p \mid \det 2T, \\ \left(\frac{-c}{p} \right) p^{k-2} C(T, \frac{1}{p} S) & \text{if } p \mid a \text{ and } p \mid \det 2T, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

$$\begin{aligned}
\alpha_{2,0,0}(T, S) &= p^{4k-8} C\left(\frac{1}{p^2}T, \frac{1}{p^2}S\right), \\
\alpha_{2,0,1}(T, S) &= p^{2k-5} \sum_{U \in R(p^2)} C\left(\begin{pmatrix} p^{-1} & \\ & p \end{pmatrix} UT^t U \begin{pmatrix} p^{-1} & \\ & p \end{pmatrix}, \frac{1}{p} \begin{pmatrix} p^{-1} & \\ & p \end{pmatrix} US\right), \\
\alpha_{2,0,2}(T, S) &= C(p^2 T, S),
\end{aligned}$$

$$\begin{aligned}
\alpha_{2,1,0}(T, S) &= p^{3k-7} \sum_{U \in R(p)} \left(\frac{-c_U}{p}\right) \\
&\quad \times C\left(\begin{pmatrix} p^{-1} & \\ & 1 \end{pmatrix} UT^t U \begin{pmatrix} p^{-1} & \\ & 1 \end{pmatrix}, \frac{1}{p} \begin{pmatrix} p^{-1} & \\ & 1 \end{pmatrix} US\right), \\
&\quad \text{where } \begin{pmatrix} * & * \\ * & c_U \end{pmatrix} = UT^t U, \\
\alpha_{2,1,1}(T, S) &= p^{k-3} \sum_{U \in R(p)} \left(\frac{-a_U}{p}\right) \\
&\quad \times C\left(\begin{pmatrix} 1 & \\ & p \end{pmatrix} UT^t U \begin{pmatrix} 1 & \\ & p \end{pmatrix}, \frac{1}{p} \begin{pmatrix} 1 & \\ & p \end{pmatrix} US\right), \\
&\quad \text{where } \begin{pmatrix} a_U & * \\ * & * \end{pmatrix} = UT^t U, \\
\alpha_{2,2,0}(T, S) &= \begin{cases} -p^{2k-6} C(T, \frac{1}{p}S) & \text{if } p \nmid \det 2T, \\ (p-1)p^{2k-6} C(T, \frac{1}{p}S) & \text{if } p \mid \det 2T. \end{cases}
\end{aligned}$$

For $p = 2$ we have the same $\alpha_{1,i,j}(T, S)$ and $\alpha_{2,i,j}(T, S)$ in the above formula by replacing the condition $p \mid \det(2T)$ or $p \nmid \det(2T)$ by $8 \mid \det(T)$ or $8 \nmid \det(T)$. (cf. [H-I 05, section 4.2].)

8.3. The function $\tilde{K}_{i,j}^{j-i}$ for $n = 2$. In this subsection we shall give some necessary data for the calculation of $\tilde{K}_{i,j}^{j-i}$, namely we need these data for the proof of Lemma 7.2.

We put

$$\begin{aligned}
L(p) &:= \left\{ \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}, \begin{pmatrix} & -1 \\ 1 & p \end{pmatrix} \mid x \pmod{p} \right\}, \\
L(p^2) &:= \left\{ \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}, \begin{pmatrix} & -1 \\ 1 & py \end{pmatrix} \mid x \pmod{p^2}, y \pmod{p} \right\}.
\end{aligned}$$

Then we have the following table:

(i, j)	$\sum_{u_2 \in (\mathbb{Z}/p\mathbb{Z})^{(j-i, 1)}} G_{\mathcal{M}}^{(j-i, 0)}(\lambda + (0, u_2))$	$L_{i,j}^*$	$\Gamma(\delta_{i,j}) \backslash \Gamma_{\infty}^{(2)}$
$(2, 2)$	1	$\begin{pmatrix} p\mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} \end{pmatrix}$	1
$(1, 1)$	1	$\begin{cases} \begin{pmatrix} p\mathbb{Z} & \mathbb{Z} \\ \frac{1}{p}\mathbb{Z} & \mathbb{Z} \end{pmatrix} & \text{if } p f \\ \begin{pmatrix} p\mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} & \text{if } p \nmid f \end{cases}$	$L(p^2)$
$(0, 0)$	1	$\begin{cases} \begin{pmatrix} \frac{1}{p}\mathbb{Z} & \mathbb{Z} \\ \frac{1}{p}\mathbb{Z} & \mathbb{Z} \end{pmatrix} & \text{if } p f \\ \mathbb{Z}^{(2,2)} & \text{if } p \nmid f \end{cases}$	1
$(1, 2)$	$\begin{cases} 0 & \text{if } \lambda \in p\mathbb{Z} \times \mathbb{Z} \\ p \left(\frac{D_0 f^2}{p} \right) & \text{if } \lambda \notin p\mathbb{Z} \times \mathbb{Z} \end{cases}$	$\begin{pmatrix} p\mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$	$L(p)$
$(0, 1)$	$\begin{cases} 0 & \text{if } \lambda \in p\mathbb{Z} \times \mathbb{Z} \\ p \left(\frac{D_0 f^2}{p} \right) & \text{if } \lambda \notin p\mathbb{Z} \times \mathbb{Z} \end{cases}$	$\begin{cases} \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \frac{1}{p}\mathbb{Z} & \mathbb{Z} \end{pmatrix} & \text{if } p f \\ \mathbb{Z}^{(2,2)} & \text{if } p \nmid f \end{cases}$	$L(p)$
$(0, 2)$	$p^2(p-1)$	$\mathbb{Z}^{(2,2)}$	1

8.4. Certain summations. In this subsection we shall give some formulas which are needed for the proof of Lemma 7.1 and 7.2. Let notation be as in §5.1 and §5.2.

Lemma 8.3. *Let F be a function on $\mathbb{Z}^{(2,2)}$. Then we obtain*

$$\sum_{\begin{pmatrix} A & B \\ 0_2 & {}^t A^{-1} \end{pmatrix} \in \Gamma(\delta_{1,2}) \backslash \Gamma_{\infty}^{(2)}} \sum_{\lambda \in L_{1,2}^*} F({}^t A \lambda) = \sum_{\lambda \in \mathbb{Z}^{(2,2)}} F(\lambda) + p \sum_{\lambda \in \mathbb{Z}^{(2,2)}} F(\lambda \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}),$$

if the above summations are absolutely convergent.

Proof. A bijection map $\Gamma(\delta_{1,2}) \backslash \Gamma_{\infty}^{(2)} \rightarrow \delta_{1,2} GL_2(\mathbb{Z}) \delta_{1,2}^{-1} \cap GL_2(\mathbb{Z}) \backslash GL_2(\mathbb{Z})$ is given via $\begin{pmatrix} A & B \\ 0_2 & {}^t A^{-1} \end{pmatrix} \mapsto A$. Thus

$$\sum_{\begin{pmatrix} A & B \\ 0_2 & {}^t A^{-1} \end{pmatrix} \in \Gamma(\delta_{1,2}) \backslash \Gamma_{\infty}^{(2)}} \sum_{\lambda \in L_{1,2}^*} F({}^t A \lambda) = \sum_{A \in \delta_{1,2} GL_2(\mathbb{Z}) \delta_{1,2}^{-1} \cap GL_2(\mathbb{Z}) \backslash GL_2(\mathbb{Z})} \sum_{\lambda \in L_{1,2}^*} F({}^t A \lambda).$$

Now we have $L_{1,2}^* = \left\{ \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \in \mathbb{Z}^{(2,2)} \mid \lambda_1 \in p\mathbb{Z} \times \mathbb{Z}, \lambda_2 \in \mathbb{Z} \times \mathbb{Z} \right\}$. We define

$$L(p) := \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & p \end{pmatrix} \mid x \pmod{p} \right\}.$$

Then $L(p)$ is a complete set of representatives of $\delta_{1,2} GL_2(\mathbb{Z}) \delta_{1,2}^{-1} \cap GL_2(\mathbb{Z}) \backslash GL_2(\mathbb{Z})$. For $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{Z}^{(2,1)}$, we define $S_{\begin{pmatrix} a \\ b \end{pmatrix}} := \left\{ A \in L(p) \mid {}^t A^{-1} \begin{pmatrix} a \\ b \end{pmatrix} \in L_{1,2}^* \right\}$. We obtain

$$\#S_{\begin{pmatrix} a \\ b \end{pmatrix}} = \begin{cases} p+1 & \text{if } a \equiv b \equiv 0 \pmod{p}, \\ 1 & \text{otherwise.} \end{cases}$$

Hence

$$\begin{aligned}
& \sum_{A \in \delta_{1,2} GL_2(\mathbb{Z}) \delta_{1,2}^{-1} \cap GL_2(\mathbb{Z}) \backslash GL_2(\mathbb{Z})} \sum_{\lambda \in L_{1,2}^*} F({}^t A \lambda) \\
&= (p+1) \sum_{\lambda \in \begin{pmatrix} p\mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} \end{pmatrix}} F(\lambda) + \sum_{\lambda \in \mathbb{Z}^{(2,2)} \backslash \begin{pmatrix} p\mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} \end{pmatrix}} F(\lambda) \\
&= \sum_{\lambda \in \mathbb{Z}^{(2,2)}} F(\lambda) + p \sum_{\lambda \in \mathbb{Z}^{(2,2)}} F(\lambda \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}).
\end{aligned}$$

□

Lemma 8.4. *Let F be a function on $\mathbb{Z}^{(2,2)}$. Then we obtain*

$$\begin{aligned}
& \sum_{\begin{pmatrix} A & B \\ 0_2 & {}^t A^{-1} \end{pmatrix} \in \Gamma(\delta_{1,1}) \backslash \Gamma_\infty^{(2)}} \sum_{\lambda \in \begin{pmatrix} p^2\mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}} F({}^t A \lambda) \\
&= \sum_{\lambda \in \mathbb{Z}^{(2,2)}} F(\lambda) + (p-1) \sum_{\lambda \in \mathbb{Z}^{(2,2)}} F(\lambda \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) + p^2 \sum_{\lambda \in \mathbb{Z}^{(2,2)}} F(\lambda \begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix}),
\end{aligned}$$

if the above summations are absolutely convergent.

Proof. We have

$$\begin{aligned}
& \sum_{\begin{pmatrix} A & B \\ 0_2 & {}^t A^{-1} \end{pmatrix} \in \Gamma(\delta_{1,1}) \backslash \Gamma_\infty^{(2)}} \sum_{\lambda \in \begin{pmatrix} p^2\mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}} F({}^t A \lambda) \\
&= \sum_{A \in \delta_{1,1} GL_2(\mathbb{Z}) \delta_{1,1}^{-1} \cap GL_2(\mathbb{Z}) \backslash GL_2(\mathbb{Z})} \sum_{\lambda \in \begin{pmatrix} p^2\mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}} F({}^t A \lambda).
\end{aligned}$$

We define $L(p^2) := \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & py \end{pmatrix} \mid x \pmod{p^2}, y \pmod{p} \right\}$. Then $L(p^2)$ is a complete set of representatives of $\delta_{1,1} GL_2(\mathbb{Z}) \delta_{1,1}^{-1} \cap GL_2(\mathbb{Z}) \backslash GL_2(\mathbb{Z})$. For $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{Z}^{(2,1)}$, we define $S_{\begin{pmatrix} a \\ b \end{pmatrix}} := \left\{ A \in L(p^2) \mid {}^t A^{-1} \begin{pmatrix} a \\ b \end{pmatrix} \in \begin{pmatrix} p^2\mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} \right\}$. By a straightforward calculation we obtain

$$\#S_{\begin{pmatrix} a \\ b \end{pmatrix}} = \begin{cases} p^2 + p & \text{if } p^2 \mid a \text{ and } p^2 \mid b, \\ 1 & \text{if } p \nmid a \text{ or } p \nmid b, \\ p & \text{otherwise.} \end{cases}$$

Hence we get

$$\begin{aligned}
& \sum_{A \in \delta_{1,1} GL_2(\mathbb{Z}) \delta_{1,1}^{-1} \cap GL_2(\mathbb{Z}) \setminus GL_2(\mathbb{Z})} \sum_{\lambda \in \begin{pmatrix} p^2 \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}} F({}^t A \lambda) \\
&= \sum_{\lambda \in \mathbb{Z}^{(2,2)}} F(\lambda) + (p-1) \sum_{\lambda \in \begin{pmatrix} p \mathbb{Z} & \mathbb{Z} \\ p \mathbb{Z} & \mathbb{Z} \end{pmatrix}} F(\lambda) + p^2 \sum_{\lambda \in \begin{pmatrix} p^2 \mathbb{Z} & \mathbb{Z} \\ p^2 \mathbb{Z} & \mathbb{Z} \end{pmatrix}} F(\lambda).
\end{aligned}$$

□

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