Zeta function and Zharkovskaya's theorem on half integral weights Siegel modular forms

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Introduction

The relation between the spinor L-function of Siegel modular forms of integral weights and Siegel Φ operator was studied by Zharkovskaya [8]. She showed the commuting relation between the Siegel Φ operator and the Hecke operators acting on the space of Siegel modular forms of integral weights, moreover she showed that the homomorphic map from Hecke ring of degree n to of degree $n-1$ is surjective, consequently she showed that the quotient part of the spinor L-function of Siegel modular form F of degree n was written by using the quotient part of the spinor L-function of Siegel modular form $\Phi(F)$ of degree $n-1$ where $\Phi(F)$ is the image of F by Siegel Φ operator. This Zharkovskaya's theorem was generalized for arbitrary level by Andrianov [1].

The even zeta function of Siegel modular forms of *half integral weights* was studied by Zhuravlev [6] [7]. It was also known by Oh-Koo-Kim [4] that the commuting relation between the Siegel Φ operator and Hecke operators acting on the space of Siegel modular forms of *half integral weights* , and also they showed that the map from suitable Hecke ring of degree n to of degree $n-1$ is surjective.

In this article we showed the relation between even zeta functions of Siegel modular forms of *half integral weights* of degree n and of degree $n-1$ (Theorem 2). As attention, this theorem 2 has been already treated in the case degree $n = 2$, level $q = 4$ and character $\chi \equiv 1$, by Hayashida-Ibukiyama [3], but in this article main result is deduced from theorem of Oh-Koo-Kim[4].

Notation We let $\mathbb{Z}$,$\mathbb{Q}$,$\mathbb{R}$ and $\mathbb{C}$ denote the ring of rational integers, the field of rational numbers, the field of real numbers, and the field of complex numbers. Let $M_{m,n}(A)$ be the set of all $m \times n$ matrices over a commutative ring with unit A, and we put $M_n(A) = M_{n,n}(A)$. For matrices $N \in M_n(A)$ and $M \in M_{n,m}(A)$, we define $N[M] = {}^tMNM$ where ${}^tM$ is the transpose of M. We put $M^* = {}^tM^{-1}$. Let E_n be the identity matrix and let $GL_n(A)$ be the group of invertible matrices in $M_n(A)$ and $SL_n(A)$ the subgroup consisting of matrices with determinant 1. If $A \subset \mathbb{R}$, and $A_+^\times$ is the group of positive units of A, then we put

$$GSp_n^+(A) = \left\{ M \in M_{2n}(A) \,|\, {}^tMJ_nM = \gamma(M)J_n \ , \ \gamma(M) \in A_+^\times \right\} ,$$

where $J_n = \begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix}$. We denote positive determinant matrices in $M_n(A)$ by $M_n^+(A)$. We define $Sp_n(A)$ as follows,

$$Sp_n(A) = \left\{ M \in GSp_n^+(A) \,|\, \gamma(M) = 1 \right\} .$$

We put $e(M) = \exp(2\pi i\sigma(M))$, $\sigma(M)$ is the trace of the matrix M.

$$\mathfrak{Z}_n \ = \ \left\{ Z = X + iY \in M_n(\mathbb{C}) \mid Z = {}^tZ, \, Y > 0 \right\}$$

is the Siegel upper half-space of degree n. We denote the action of $Sp_n(\mathbb{R})$ on $\mathfrak{Z}_n$ by

$$M\langle Z\rangle \ = \ (AZ + B)(CZ + D)^{-1}$$

where $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp_n(\mathbb{R})$, and $Z \in \mathfrak{Z}_n$.

For positive integer q, we put

$$\Gamma_0^n(q) = \{ M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp_n(\mathbb{Z}) \mid C \equiv 0 \, (mod \, q) \}$$

is the congruence-subgroup of the symplectic group $Sp_n(\mathbb{Z})$.

We set $(a, b) = \gcd(a, b)$ for integers a and b. We let $\langle n\rangle = n(n + 1)/2$ for $n \in \mathbb{Z}$.

1 Hecke rings

The Hecke ring $\hat{L}_p^n(\kappa)$ was introduced by Zhuravlev [6], [7] to interpret the even zeta function of Siegel modular forms of *half integral weights* for general degree. The aim of this section is to describe this Hecke ring $\hat{L}_p^n(\kappa)$ according to Zhuravlev [6], [7].

1.1 Hecke pair and Hecke ring

Let Γ be a group and let S be a semigroup in the multiplicative group G. (Γ, S) is called a Hecke pair if $\Gamma S = S\Gamma = S$ and if for any $g \in S$ the quotient sets $\Gamma\backslash\Gamma g\Gamma$ and $\Gamma g\Gamma/\Gamma$ are finite. Let $L(\Gamma,S)$ be the $\mathbb{C}$-module spanned by the left cosets (Γg), $g\in S$. By the Hecke ring $D(\Gamma,S)$ we mean the Γ-invariant submodule of $L(\Gamma,S)$ consisting of $X=\sum_i a_i(\Gamma g_i)$ such that $X\cdot\gamma=X$ for any $\gamma\in\Gamma$, where $X\cdot\gamma=\sum_i a_i(\Gamma g_i\gamma)$. If $X=\sum_i a_i(\Gamma g_i)$ and $Y=\sum_j b_j(\Gamma h_j)$ in $D(\Gamma,S)$, then by definition, $X\cdot Y=\sum_{i,j}a_ib_j(\Gamma g_ih_j)$. $D(\Gamma,S)$ is an associative ring with generators $(\Gamma g\Gamma)=\sum_i(\Gamma g_i)$, where $g\in S$ and $\Gamma g\Gamma=\cup_i\Gamma g_i$ is the left coset decomposition.

We put $\Gamma_0^n(q)$ and Γ_0^n, these are subgroup of $Sp(n,\mathbb{Z})$, as follows,

$$\begin{aligned}\Gamma_0^n(q) &= \{M=\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)\in Sp(n,\mathbb{Z})\mid C\equiv 0\,(q)\},\\ \Gamma_0^n &= \{M=\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)\in Sp(n,\mathbb{Z})\mid C=0\}.\end{aligned}$$

We denote $\mathbb{Z}[p^{-1}]$ by $\mathbb{Z}[p^{-1}]=\left\{\frac{a}{p^r}\in\mathbb{Q}\mid a,r\in\mathbb{Z}\right\}$. We put S_p^n, $S_{p^2}^n$ and $S_{0,p}$, these are multiplicative set in $GSp_n^+(\mathbb{Z}[p^{-1}])$, as follows

$$\begin{aligned}S_p^n &= \{M=\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)\in GSp_n^+(\mathbb{Z}[p^{-1}])\mid C\equiv 0\,(q)\,,\,\gamma(M)=p^{\delta}\},\\ S_{p^2}^n &= \{M=\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)\in GSp_n^+(\mathbb{Z}[p^{-1}])\mid C\equiv 0\,(q)\,,\,\gamma(M)=p^{2\delta}\},\\ S_{0,p}^n &= \{M=\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)\in S_p^n\mid C=0\}.\end{aligned}$$

We put $\Lambda_n=SL(n,\mathbb{Z})$ and $G_p^n=\{D\in M_n(\mathbb{Z})\mid \det D=p^{\delta},\delta=0,1,2,...\}$.

It was known that $(\Gamma_0^n(q),S_p^n)$, $(\Gamma_0^n(q),S_{p^2}^n)$, $(\Gamma_0^n,S_{0,p}^n)$ and (Λ_n,G_p^n) are Hecke pairs. We shall denote the corresponding Hecke rings by $L_p^n(q)=D(\Gamma_0^n(q),S_p^n)$, $L_{p^2}^n(q)=D(\Gamma_0^n(q),S_{p^2}^n)$, $L_{0,p}^n=D(\Gamma_0^n,S_{0,p}^n)$ and $H_p^n=D(\Lambda_n,G_p^n)$.

1.2 universal covering group

The universal covering group $\mathfrak{G}$ for $GSp_n^+(\mathbb{R})$ consists of pairs $(M,\varphi(Z))$, where $M=\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)$ is in $GSp_n^+(\mathbb{R})$, $\varphi(Z)$ is a holomorphic function on $\mathfrak{Z}_n$ and $|\varphi(Z)|^2=\det M^{-1/2}|\det(CZ+D)|$, with the group operation

$$(M,\varphi(Z))\cdot(L,\psi(Z)) \;=\; (ML,\varphi(L\langle Z\rangle)\psi(Z)).$$

We define the standard theta series and a function j

$$\begin{aligned}\Theta^n(Z) &= \sum_{m\in\mathbb{Z}^n}e(Z[m]), \qquad (Z\in\mathfrak{Z}_n)\\ j(M,Z) &= \frac{\Theta^n(M\langle Z\rangle)}{\Theta^n(Z)}, \qquad (M\in\Gamma_0^n(4),Z\in\mathfrak{Z}_n)\end{aligned}$$

We define an injective homomorphism $j;\Gamma_0^n(q)\to\mathfrak{G}$ by setting $j(M)=(M,j(M,Z))$. We put subgroups $\hat{\Gamma}_0^n(q)=j(\Gamma_0^n(q))$, $\hat{\Gamma}_0^n=j(\Gamma_0^n)$, these are subgroups of $\mathfrak{G}$.

We define projection $P : \mathfrak{G} \ni (M, \varphi(Z)) \to M \in GSp_n^+(\mathbb{R})$ and we put $\hat{S}_p^n = P^{-1}(S_p^n)$, $\hat{S}_{p^2}^n = P^{-1}(S_{p^2}^n)$, $\hat{S}_{0,p}^n = P^{-1}(S_{0,p}^n)$. It is known that $(\hat{\Gamma}_0^n(q), \hat{S}_p^n)$, $(\hat{\Gamma}_0^n(q), \hat{S}_{p^2}^n)$ and $(\hat{\Gamma}_0^n, \hat{S}_{0,p}^n)$ are also Hecke pairs. We put Hecke rings $\hat{L}_p^n(q) = D(\hat{\Gamma}_0^n(q), \hat{S}_p^n)$, $\hat{L}_{p^2}^n(q) = D(\hat{\Gamma}_0^n(q), \hat{S}_{p^2}^n)$ and $\hat{L}_{0,p}^n = D(\hat{\Gamma}_0^n, \hat{S}_{0,p}^n)$.

1.3 The reduction of the Hecke ring

We define homomorphism $\hat{\varepsilon}_{q,0} : \hat{L}_p^n(q) \to \hat{L}_{0,p}^n$, as follows; for any element $\xi \in \hat{S}_p^n$, there exist some elements $\gamma = \hat{\Gamma}_0^n(q)$ and $\xi_0 \in \hat{S}_0^n$ such that $\xi = \gamma\xi_0$, we define

$$\hat{\varepsilon}_{q,0}(\hat{\Gamma}_0^n(q)\xi\hat{\Gamma}_0^n(q)) = \hat{\Gamma}_0^n\xi_0\hat{\Gamma}_0^n.$$

For odd integer $2k-1$, we define homomorphism $P_{2k-1} : \hat{L}_{0,p}^n \to L_{0,p}^n$ by

$$P_{2k-1}(\hat{\Gamma}_0^n\xi_0\hat{\Gamma}_0^n) = \left(\frac{\varphi(Z)}{|\varphi(Z)|}\right)^{-2k+1} (\Gamma_0^n M \Gamma_0^n),$$

where $\xi_0 = (M, \varphi(Z)) \in \hat{S}_{0,p}^n$ and the function $\varphi(Z)|\varphi(Z)|^{-1}$ does not depend on the choice of representative element ξ_0 because of the definition $\hat{\Gamma}_0^n$, moreover $\varphi(Z)|\varphi(Z)|^{-1}$ is constant function on $Z \in \mathfrak{Z}_n$ because of the definition of $\hat{S}_{0,p}^n$,

The surjective homomorphism $\Omega_n; L_{0,p}^n \to H_p^n[t^{\pm 1}]$, where t is transcendental over H_p^n, is defined as follows; for $X \in L_{0,p}^n$ written in the form $X = \sum_i a_i \left(\Gamma_0^n \begin{pmatrix} p^{\delta_i} D_i^* & B_i \\ 0 & D_i \end{pmatrix}\right)$, we set

$$\Omega_n(X) = \sum_i a_i t^{\delta_i}(\Lambda_n D_i).$$

Let $x_0, ..., x_n$ be algebraically independent over $\mathbb{C}$, let $h = \sum_i a_i t^{\delta_i}(\Lambda_n D_i)$ be in $H_p^n[t^{\pm 1}]$, and suppose that for D_i we take upper triangular matrices with diagonal elements $p^{d_{i1}}, ..., p^{d_{in}}$. Then we define the injective homomorphism $\varphi : H_p^n[t^{\pm 1}] \to \mathbb{C}[x_0^{\pm 1}, ..., x_n^{\pm 1}]$ by setting

$$\varphi(h) = \sum_i a_i x_0^{\delta_i} \prod_{j=1}^{n} (x_j p^{-j})^{d_{ij}}.$$

If it collects, the above maps are as follows,

$$\hat{L}_p^n(q) \overset{\hat{\varepsilon}_{q,0}}{\to} \hat{L}_{0,p}^n \overset{P_{2k-1}}{\to} L_{0,p}^n \overset{\Omega_n}{\to} H_p^n[t^{\pm 1}] \overset{\varphi}{\to} \mathbb{C}[x_0^{\pm 1}, ..., x_n^{\pm 1}],$$

1.4 Hecke ring $\hat{L}_p^n(\kappa)$

We consider the commutative subring $\hat{L}_p^n(\kappa)$ $(\subset \hat{L}_{p^2}^n(q))$ which is generated over $\mathbb{C}$ by the elements $\hat{T}(K_0), ..., \hat{T}(K_{n-1}), \hat{T}(K_n)^{\pm 1}$, where $\hat{T}(K_s) = (\hat{\Gamma}_0^n(q)\hat{K}_s\hat{\Gamma}_0^n(q))$, $K_s = diag(E_{n-s}, pE_s; p^2E_{n-s}, pE_s)$ and $\hat{K}_s = (K_s, p^{(n-s)/2})$ are the corresponding elements of $\mathfrak{G}$. We define $\mathbf{L}_p^n(\kappa) = P_{2k-1}\hat{\varepsilon}_{q,0}(\hat{L}_p^n(\kappa))$. Let $\mathbb{C}^{W_2}[x_0^{\pm 1}, ..., x_n^{\pm 1}]$ be the ring of W_2-invariant polynomials, where W_2 is the automorphism group generated by the permutations of $x_1, ..., x_n$, the transformations $x_0 \to x_0x_i, x_i \to x_i^{-1}, x_j \to x_j (j \neq 0, i; i = 1, ..., n)$, and the transformation $x_0 \to -x_0, x_i \to x_i (i \neq 0)$. Then the homomorphism map $\varphi \circ \Omega_n$ gives an isomorphism of $\mathbf{L}_p^n(\kappa)$ with the respective polynomial ring $\mathbb{C}^{W_2}[x_0^{\pm 1}, ..., x_n^{\pm 1}]$. (see Zhuravlev [7])

Namely, there exist the isomorphisms as follows,

$$\hat{L}_p^n(\kappa) \simeq \mathbf{L}_p^n(\kappa) \simeq \mathbb{C}^{W_2}[x_0^{\pm 1}, ..., x_n^{\pm 1}]. \tag{1}$$

2 Siegel modular forms of half integral weights and Hecke operators

In the theory of modular forms, the Hecke rings give a representation on the space of modular forms, and this representation is important to consider the multiplicative property of Fourier coefficients of modular forms. In this section we describe the representation of Hecke rings on the space of Siegel modular forms of *half integral weights* according to Zhuravlev [7].

2.1 Siegel modular forms of half integral-weights

Let k be an integer, let χ be a Dirichlet character modulo q, and let $4|q$. Then a holomorphic function $F(Z)$ on $\mathfrak{Z}_n$ is said to be a *Siegel modular form* of weight $k - 1/2$ and character χ belongs to $\Gamma_0^n(q)$ if

$$F(M\langle Z\rangle) = \chi(\det D)j(M, Z)^{2k-1}F(Z)\,, \text{ for any } M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_0^n(q),$$

and in the case $n = 1$ the function $F(Z)$ is holomorphic at all cusps of $\Gamma_0^1(q)$. We denote the set of such functions by $\mathfrak{M}_{k-1/2}^n(q, \chi)$. If $n = 0$ then we denote $\mathfrak{M}_{k-1/2}^0(q, \chi) = \mathbb{C}$ for $k > 0$. Siegel modular forms have a Fourier expansion

$$F(Z) = \sum_{N \in \mathfrak{N}_n} f(N)e(NZ),$$

where $\mathfrak{N}_n$ is the set of symmetric positive semi-definite half-integral matrices of size n. From the definition of $\mathfrak{M}^n_{k-1/2}(q,\chi)$ it follows that $f(N[U]) = f(N)$ for $U \in SL_n(\mathbb{Z})$.

For any function $F(Z)$ on $\mathfrak{Z}_n$ and for $\xi = (M, \varphi(Z)) \in \hat{\Gamma}^n_0(q)$ we set

$$F|_{k-1/2,\chi}\xi \;\; = \;\; \gamma(M)^{n(2k-1)/4-\langle n\rangle}\chi(\det A)\varphi(Z)^{-2k+1}F(M\langle Z\rangle).$$

It follows from the definition that $F|_{k-1/2,\chi}\xi_1|_{k-1/2,\chi}\xi_2 = F|_{k-1/2,\chi}\xi_1\xi_2$, and, if $F \in \mathfrak{M}^n_{k-1/2}(q,\chi)$, then $F|_{k-1/2,\chi}\xi = F$ for any $\xi \in \hat{\Gamma}^n_0(q)$.

2.2 Representations of Hecke rings on Siegel modular forms of half-integral weights.

For $F \in \mathfrak{M}^n_{k-1/2}(q,\chi)$, we define representation of the Hecke ring $\hat{L}^n_{p^2}(q)$ by setting

$$F|_{k-1/2,\chi}\hat{X} \;\; = \;\; \sum_i a_i F|_{k-1/2,\chi}\hat{M}_i$$

where $\hat{X} = \sum_i a_i(\hat{\Gamma}^n_0(q)\hat{M}_i) \in \hat{L}^n_{p^2}(q)$. We define representation of the Hecke ring $L^n_{0,p}$ by setting

$$F|_{k-1/2,\chi}X \;\; = \;\; \sum_j b_j F|_{k-1/2,\chi}M_j$$

where $X = \sum_j b_j(\Gamma^n_0 M_j) \in L^n_{0,p}$, and where

$$F|_{k-1/2,\chi}M = F|_{k-1/2,\chi}\hat{M} \quad \textit{and} \quad \hat{M} = (M, \gamma(M)^{-n/4}|\det D|^{1/2})$$

for a matrix $M = \left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right) \in GSp^+_n(\mathbb{Q})$.

The following equation was shown by Zhuravlev [7]; for any $F \in \mathfrak{M}^n_{k-1/2}(q,\chi)$ and any $\hat{X} \in \hat{L}^n_{p^2}(q)$, we have

$$F|_{k-1/2,\chi}\hat{X} \;\; = \;\; F|_{k-1/2,\chi}P_{2k-1}\hat{\varepsilon}_{q,0}(\hat{X}). \tag{2}$$

By virtue of the above equation, we can consider the action of Hecke ring $\hat{L}^n_{p^2}(q)$ on Siegel modular forms $\mathfrak{M}^n_{k-1/2}(q,\chi)$ as the action of corresponding Hecke ring in $L^n_{0,p}$.

3 The Ψ operator and Siegel Φ-operator

The Ψ operator was introduced by Andrianov [1] to generalize the theorem of Zharkovskaya [8] for arbitrary level. This Ψ operator was also considered for Siegel modular forms of *half integral weights* by Oh-Koo-Kim [4]. In this section we review this Ψ operator , and introduce the theorem of Oh-Koo-Kim[4].

Let $X = \sum_i a_i(\Gamma_0^n g_i) \in L_{0,p}^n$ where $g_i = \begin{pmatrix} p^{\delta_i} D_i^* & B_i \\ 0 & D_i \end{pmatrix}$. We can take this D_i upper triangular and set $D_i = \begin{pmatrix} D_i^{'} & * \\ 0 & p^{d_i} \end{pmatrix}$, $D_i^{'}$ is also upper triangular. We set homomorphism $\Psi(X,u) : L_{0,p}^n \to L_{0,p}^{n-1}[u^{\pm 1}]$ by

$$\Psi(X,u) \;=\; \sum_i a_i u^{-\delta_i}(up^{-n})^{d_i}(\Gamma_0^{n-1} g_i^{'}),$$

where u is an independent variable and $g_i^{'} = \begin{pmatrix} p^{\delta_i} D_i^{'*} & B_i^{'} \\ 0 & D_i^{'} \end{pmatrix}$, $B_i^{'}$ denotes the block of size $n-1$ in the upper left corner of B_i. If $n=1$, we set $\Psi(X,u) = \sum_i a_i u^{-\delta_i}(up^{-1})^{d_i}$.

We define $\mathbb{C}$-linear homomorphism $\eta_{n,u} : \mathbb{C}[x_0^{\pm 1}, ..., x_n^{\pm 1}] \to \mathbb{C}[x_0^{\pm 1}, ..., x_{n-1}^{\pm 1}, u^{\pm 1}]$ by following condition

$$\eta_{n,u}(x_0) = x_0 u^{-1}\ ,\ \eta_{n,u}(x_n) = u\ ,\ \eta_{n,u}(x_i) = x_i\ (i = 1, ..., n-1)$$

then the following diagram is commutative :

$$\begin{array}{ccc} L_{0,p}^n & \overset{\varphi\circ\Omega_n}{\longrightarrow} & \mathbb{C}[x_0^{\pm 1}, ..., x_n^{\pm 1}] \\ {\scriptstyle \Psi(-,u)} \downarrow & \circlearrowleft & \eta_{n,u} \downarrow \\ L_{0,p}^{n-1}[u^{\pm 1}] & \overset{\varphi\circ\Omega_{n-1}\times 1}{\longrightarrow} & \mathbb{C}[x_0^{\pm 1}, ..., x_{n-1}^{\pm 1}, u^{\pm 1}] \end{array} \tag{3}$$

where $\varphi \circ \Omega_{n-1} \times 1$ is the ring homomorphism defined by $\varphi \circ \Omega_{n-1} \times 1 = \varphi \circ \Omega_{n-1}$ on $L_{0,p}^{n-1}$, $(\varphi \circ \Omega_{n-1} \times 1)(u^{\pm 1}) = u^{\pm 1}$.

Let $F \in \mathfrak{M}_{k-1/2}^n(q,\chi)$. We define the map $\Phi : \mathfrak{M}_{k-1/2}^n(q,\chi) \to \mathfrak{M}_{k-1/2}^{n-1}(q,\chi)$ by

$$\Phi(F)(Z) \;=\; \lim_{\lambda \to +\infty} F\begin{pmatrix} Z & 0 \\ 0 & i\lambda \end{pmatrix}, \quad Z \in \mathfrak{Z}_{n-1},$$

this Φ is called Siegel Φ-operator.

The following theorem was shown by Oh-Koo-Kim [4].

Theorem 1 (Oh-Koo-Kim).
Let $F \in \mathfrak{M}_{k-1/2}^n(q,\chi)$ and $\hat{X} \in \hat{L}_{p^2}^n(q)$. Then we have

$$\Phi(F|_{k-1/2,\chi}\hat{X}) \;=\; \Phi(F)|_{k-1/2,\chi}\Psi(X, p^{n-(k-1/2)}\chi(p)^{-1}), \tag{4}$$

where $X = P_{2k-1}\hat{\varepsilon}_{q,0}(\hat{X}) \in L^n_{0,p^2}$. *(If* $n = 1$*, then the right hand side of above equation is the action of* $L^0_{0,p^2} = \mathbb{C}$ *on* $\mathfrak{M}^0_{k-1/2}(q,\chi) = \mathbb{C}$*, which is just the multiplication of complex numbers.)*

Moreover, the map $\Psi(*, p^{n-(k-1/2)}\chi(p)^{-1}) : \mathbf{L}^n_p(\kappa) \to \mathbf{L}^{n-1}_p(\kappa)$ *is a surjective ring homomorphism. If* F *is eigenfunction for the action of* $\mathbf{L}^n_p(\kappa)$ *and if* $\Phi(F)$ *is not zero function then* $\Phi(F)$ *is also eigenfunction for the action of* $\mathbf{L}^{n-1}_p(\kappa)$*.*

4 The even zeta function of half integral weights Siegel modular forms

The even zeta function of Siegel modular forms of *half integral weights* was studied by Zhuravlev [7], this is generalization of the theorem on degree 1 case of Shimura [5] for general degree. In this section we review his result according to [7].

Let $\gamma(z)$ be the polynomial defined by

$$\gamma(z) \;=\; \prod_{i=1}^{n}(1 - x_i z)(1 - x_i^{-1} z),$$

then light hand side of this equation has expansion

$$\gamma(z) \;=\; \sum_{i=0}^{2n}(-1)^i R^n_i z^i.$$

where $R^n_i \in \mathbb{C}^{W_2}[x_0^{\pm 1}, ..., x_n^{\pm 1}]$. Because of the isomorphism of $\varphi \circ \Omega_n$ (see eq(1)), there exists Hecke operator $R^n_{i,p} \in \mathbf{L}^n_p(\kappa)$ such that $\varphi \circ \Omega_n(R^n_{i,p}) = R^n_i$.

Let $F(Z) = \sum f(M)e(MZ) \in \mathfrak{M}^n_{k-1/2}(q,\chi)$ be an eigenfunction for the action of $\mathbf{L}^n_p(\kappa)$. We denote eigen values of F for Hecke operators $R^n_{i,p}$ by $\lambda_F(R^n_{i,p})$. Since $R^n_{2n-i} = R^n_i$ and $\lambda_F(R^n_{2n,p}) = 1$, we define *p-parameters* $\{\alpha^{\pm 1}_{i,p}\}$ of F as follows,

$$\prod_{i=1}^{n}(1 - \alpha_{i,p} z)(1 - \alpha^{-1}_{i,p} z) \;=\; \sum_{i=0}^{2n}(-1)^i \lambda_F(R^n_{i,p}) z^i.$$

Now we describe the result of Zhuravlev [7]. Let λ be a completely multiplicative function which grows no faster than some power of argument, and let N be positive definite matrix in $\mathfrak{N}_n$. When the real part of s is sufficiently large, The following series has Euler expansion, this series is called the even zeta function,

$$\sum_{\substack{M \in SL_n(\mathbb{Z})\backslash M_n^+(\mathbb{Z}) \\ (\det M, q)=1}} \frac{\lambda(\det M) f(N[{}^tM])}{(\det M)^{s+k-3/2}} = \prod_{p:prime} \frac{P_{F,p}(N,\lambda,p^{-s})}{Q_{F,p}(\lambda,p^{-s})}, \qquad (5)$$

where $P_{F,p}(N,\lambda,z)$ is the polynomial of z which degree is at most $2n$ (although this polynomial were written in the more explicit form in his paper, since this is not needed here, we omits), $Q_{F,p}(\lambda,z)$ is the polynomial of z which degree is $2n$. Especially $Q_{F,p}(\lambda,z)$ is not depend on the choice of N. The polynomial $Q_{F,p}(\lambda,z)$ was defined as follows,

$$Q_{F,p}(\lambda,z) = \prod_{i=0}^{n}(1-\alpha_{i,p}\chi(p)\lambda(p)z)(1-\alpha_{i,p}^{-1}\chi(p)\lambda(p)z), \qquad (6)$$

where $\alpha_{i,p}^{\pm 1}$ are the p-parameters of F.

5 Main theorem

Let F be a Siegel modular form of weight $k-1/2$ belongs to $\Gamma_0^n(q)$, where $q>0$ is an integer divisible by 4. We assume that F is an eigenfunction for the action of $\hat{L}_p^n(\kappa)$ (§1.4). Let λ be a completely multiplicative function which grows no faster than some power of argument.

We put $L(s,\lambda,F) = \prod_{(p,q)=1} Q_{F,p}(\lambda,p^{-s+k-3/2})^{-1}$ (see eq(5), eq(6)).

Then we obtain the following theorem, this is an analogy of the theorem of Zharkovskaya[8]

Theorem 2. *We assume $\Phi(F) \neq 0$, then we have*

$$L(s,\lambda,F) = L_1(s-n+1,\lambda,E_{2k-2n,\chi^2})L(s,\lambda,\Phi(F)),$$

where $L_1(s,\lambda,E_{2k-2n,\chi^2}) = \prod_{p,(p,q)=1}(1-\lambda(p)p^{-s})^{-1}(1-\lambda(p)\chi(p)^2p^{2k-2n-1-s})^{-1}$.

If $k>n+1$ then $L_1(s,\lambda,E_{2k-2n,\chi^2})$ is the L-function of Eisenstein series of degree 1 of weight $2k-2n$ with character χ^2 twisted by λ.

proof.

We define $R_p^n(z) \in \mathbf{L}_p^n(\kappa)[z]$ by

$$R_p^n(z) = \sum_{i=0}^{2n}(-1)^i R_{i,p}^n z^i,$$

where $R_{i,p}^n$ are elements of $\mathbf{L}_p^n(\kappa)$ defined in §4.

By using compatibility (3) and eq(4), we have

$$\begin{aligned}\Phi(F|_{k-1/2,\chi}R_p^n(z)) &= \Phi(F)|_{k-1/2,\chi}\Psi(R_p^n(z), p^{n-(k-1/2)}\chi(p)^{-1}) \\ &= (1-p^{n-(k-1/2)}\chi(p)^{-1}z)(1-p^{(k-1/2)-n}\chi(p)z) \\ &\quad \times \left(\Phi(F)|_{k-1/2,\chi}R_p^{n-1}(z)\right),\end{aligned}$$

besides,

$$\Phi(F|_{k-1/2,\chi}R_p^n(z)) = \left(\prod_{i=1}^{n}(1-\alpha_{i,p}^{-1}z)(1-\alpha_{i,p}z)\right)\Phi(F),$$

where $\alpha_{i,p}^{\pm}$ are p-parameters of F. From above we can take $\alpha_{n,p}^{\pm 1}$ by $\alpha_{n,p}^{\pm 1} = \left(\chi(p)^{-1}p^{n-(k-1/2)}\right)^{\pm 1}$, and we can regard $\alpha_{i,p}^{\pm 1}$ $(i = 1,2,...,n-1)$ as p-parameters of $\Phi(F)$.

Above all, we have

$$\begin{aligned}Q_{F,p}(\lambda, p^{-s+k-3/2}) &= (1-\lambda(p)p^{-s+n-1})\,(1-\lambda(p)\chi(p)^2p^{-s+2k-n-2}) \\ &\quad \times Q_{\Phi(F),p}(\lambda, p^{-s+k-3/2}).\end{aligned}$$

Consequently we proved theorem 2.

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