Skew-holomorphic Jacobi forms of index 1 and Siegel modular forms of half integral weight

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Abstract

The isomorphism between Kohnen's plus space and Jacobi forms of index 1 was given by Eichler-Zagier. In this article we generalize this isomorphism for higher degree in the case of skew-holomorphic Jacobi forms.

keywords : Jacobi forms, half-integral weight

1 Introduction

Kohnen [8] introduced a certain subspace of modular forms of half integral weight in order to describe Shimura correspondence more precisely, namely he defined a subspace that corresponds to modular forms of integral weight belonging to $SL(2,\mathbb{Z})$. That subspace was named Kohnen's plus space. For any *even* integer k, it was known that Kohnen's plus space of weight $k-1/2$ is linearly isomorphic to Jacobi forms of index 1 of weight k as Hecke algebra module. This fact was shown by Eichler-Zagier [4], and this isomorphism was generalized for higher degree by Ibukiyama [6]. Also, this isomorphism was generalized in the case of plus space of odd weight with character by Hayashida-Ibukiyama [5].

On the other hand, Skoruppa [9] introduced the skew-holomorphic Jacobi forms of degree 1. For *odd* integer k, he proved that the skew-holomorphic Jacobi forms of weight k are linearly isomorphic to Kohnen's plus space of weight $k-1/2$. Arakawa [3] defined skew-holomorphic Jacobi forms for general degree and he expected that the above isomorphism is valid also for higher degree.

The aim of this paper is to generalize the linear isomorphism between a certain subspace of Siegel modular forms of weight $k-1/2$ and skew-holomorphic Jacobi forms of weight k of index 1 for general degree. More precisely, the space of skew-holomorphic Jacobi forms of *odd* (resp. *even*) weight k of index 1 of degree n is linearly isomorphic to a certain subspace of Siegel modular forms of weight $k-1/2$ *without character* (resp. *with character*) of degree n. This space is a generalization of the usual plus space in Kohnen [8], Ibukiyama [6]. Moreover, we shall show that this isomorphism commutes with Hecke operators of both spaces as in [8], [6].

2 The plus space and the skew-holomorphic Jacobi forms of index 1.

In this section we review the definition of skew-holomorphic Jacobi forms and define a plus space. Then we shall give a linear isomorphism between skew-holomorphic Jacobi forms and the plus space in Theorem 1. This isomorphism induces a bijection between cusp forms of skew-holomorphic Jacobi forms and cusp forms of the plus space. We will also show that this isomorphism commutes with the action of

Hecke operators (cf. Theorem 2).

2.1 Linear isomorphisms.

For any ring K, we denote by $M_{n,m}(K)$ the set of $n \times m$ matrices with entries in K, and write $M_n(K) = M_{n,n}(K)$. We denote by $Sym(n, K)$ the set of $n \times n$ symmetric matrix with entries in K. For any natural number n, we denote by $\mathcal{H}_n$ the Siegel upper half space of degree n,

$$\begin{aligned} \mathcal{H}_n &= \{X + iY \in M_n(\mathbb{C});\ X, Y \in Sym(n, \mathbb{R}), Y > 0 \\ &\qquad (Y : \text{positive definite})\} . \end{aligned}$$

We denote by $Sp(n, \mathbb{R})$ the usual real symplectic group of size *2n*,

$$Sp(n, \mathbb{R}) = \left\{ M \in M_{2n}(\mathbb{R});\ M J_n {}^t M = J_n \right\} ,$$

where $J_n = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$ and 1_n is identity matrix of size n. The group $Sp(n, \mathbb{R})$ acts on $\mathcal{H}_n \times \mathbb{C}^n$ by

$$\begin{aligned} M(\tau, z) &= (M\tau, {}^t(c\tau + d)^{-1} z) \\ &= ((a\tau + b)(c\tau + d)^{-1}, {}^t(c\tau + d)^{-1} \tau) \end{aligned}$$

for

$$(\tau, z) \in \mathcal{H}_n \times \mathbb{C}^n, \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(n, \mathbb{R}) \quad (a, b, c, d \in M_n(\mathbb{R})) .$$

The skew-holomorphic Jacobi forms were first introduced by Skoruppa [9] in degree one case, and defined for general degree with matrix index by Arakawa [3].

We write $e(*) = e^{2\pi i *}$, and denote by $Sym^*(n, \mathbb{Z})$ the set of all $n \times n$ half integral symmetric matrices.

Definition 1 (Skoruppa [9], Arakawa [3]).
Let k be a natural number and let $F(\tau, z)$ be a function on $(\tau, z) \in \mathcal{H}_n \times \mathbb{C}^n$ which is real analytic in the real part and imaginary part of $\tau \in \mathcal{H}_n$ and holomorphic in $z \in \mathbb{C}^n$. When F satisfies the next conditions (1), (2), and (3), we say that F is a skew-holomorphic Jacobi form of weight k with index 1 .

(1) $F(\tau, z + \tau x + y) = e(-({}^t x \tau x + 2 {}^t x z)) F(\tau, z)$ *for all column vectors $x, y \in \mathbb{Z}^n$,*

(2) *$F|_k M = F$ for all $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(n,\mathbb{Z})$, where*

$$\begin{aligned}(F|_k M)(\tau, z) &= F(M\tau, {}^t(c\tau+d)^{-1}z)\,\overline{\det(c\tau+d)}^{1-k}|\det(c\tau+d)|^{-1} \\ &\quad \times e(-{}^t z(c\tau+d)^{-1}cz),\end{aligned}$$

where $\overline{\alpha}$ means the complex conjugate of $\alpha \in \mathbb{C}$.

(3) *F has a Fourier expansion of the following form.*

$$F(\tau, z) = \sum_N \sum_r A(N,r)e(tr(N\tau - \frac{1}{2}i(4N - r{}^t r)Y))e({}^t rz) \ ,$$

where Y is the imaginary part of τ, N runs over all elements in $Sym^*(n,\mathbb{Z})$, and r runs over all elements in $\mathbb{Z}^n$ satisfying $4N - r{}^t r \le 0$.

Moreover, if the Fourier coefficients $A(N,r)$ are zero unless $4N - r{}^t r < 0$, then we say that F is a skew-holomorphic Jacobi cusp form.

We denote by $J^+_{k,1} = J^{+(n)}_{k,1}$ (resp. $J^{+\,cusp}_{k,1} = J^{+\,cusp(n)}_{k,1}$) the whole space of skew-holomorphic Jacobi forms (resp. skew-holomorphic Jacobi cusp forms).

For any column vectors $m', m'' \in \mathbb{Z}^n$, we define theta functions $\theta_m(\tau, z) = \theta_{m',m''}(\tau, z)$ of characteristic $m = ({}^t m', {}^t m'')$ by

$$\begin{aligned}&\theta_{m',m''}(\tau, z) \\ &= \sum_{p \in \mathbb{Z}^n} e\left(\frac{1}{2}{}^t\left(p + \frac{m'}{2}\right)\tau\left(p + \frac{m'}{2}\right) + {}^t\left(p + \frac{m'}{2}\right)\left(z + \frac{m''}{2}\right)\right) .\end{aligned}$$

For each vector $\mu \in \mathbb{Z}^n$, we set

$$\vartheta_\mu(\tau, z) \ = \ \theta_{\mu,0}(2\tau, 2z).$$

This function $\vartheta_\mu(\tau, z)$ depends only on μ modulo 2. It is known that Jacobi forms can be written as a linear combination of these theta functions, where coefficients are functions on $\mathcal{H}_n$ (cf. Eichler-Zagier [4], Ibukiyama [6]). We show that the same results hold also for skew-holomorphic Jacobi forms. By virtue of conditions (1) and (3) of the definition of skew-holomorphic Jacobi forms, for any $F(\tau,z) \in J^{+}_{k,1}$ and any $x \in \mathbb{Z}^n$, we get

$$\begin{aligned}
F(\tau,z) &= e\big({}^t x\tau x + 2{}^t xz\big)\ F(\tau, z+\tau x) \\
&= e\big({}^t x\tau x + 2{}^t xz\big) \sum_{N,r} A(N,r) \\
&\qquad \times e(tr(N\tau - \frac{1}{2}i(4N - r{}^t r)Y))e({}^t r(z+\tau x)) \\
&= \sum_{N,r} A(N,r) e(tr((N + \frac{1}{4}(r+2x)^t(r+2x) - \frac{1}{4}r{}^t r)\tau \\
&\qquad - \frac{1}{2}i(4N - r{}^t r)Y)) \times\ e({}^t(r+2x)z).
\end{aligned}$$

Hence, if $r \equiv r^{'} \mod 2$ and if $4N - r{}^t r = 4N^{'} - r^{'}{}^t r^{'}$ then by comparing the above equality with the original Fourier expansion of $F(\tau,z)$, we get $A(N,r) = A(N^{'},r^{'})$. Therefore, we have

$$\begin{aligned}
F(\tau,z) &= \sum_{N,r} A(N,r)\, e\bigg(tr((N\tau - \frac{1}{2}i(4N - r{}^t r)Y)) + {}^t rz\bigg) \\
&= \sum_{N,r} A(N,r)\, e\bigg(\frac{1}{4}tr(((4N - r{}^t r)\overline{\tau})\bigg)\, e\bigg(\frac{1}{4}{}^t r\tau r + {}^t rz\bigg) \\
&= \sum_{r \,mod\, 2} \sum_{N} A(N,r)\, e\bigg(\frac{1}{4}tr((4N - r{}^t r)\overline{\tau})\bigg) \\
&\quad \times \sum_{\lambda \in \mathbb{Z}^n} e\bigg({}^t\Big(\lambda + \frac{r}{2}\Big)\tau\Big(\lambda + \frac{r}{2}\Big) + 2\,{}^t\Big(\lambda + \frac{r}{2}\Big)z\bigg).
\end{aligned}$$

Hence, for each element $F \in J^{+}_{k,1}$, there exists a set of 2^n numbers of anti-holomorphic functions $F_\mu(\tau)\,(\mu \in (\mathbb{Z}/2\mathbb{Z})^n)$ on $\mathcal{H}_n$ satisfying

$$F(\tau,z) = \sum_{\mu \in (\mathbb{Z}/2\mathbb{Z})^n} F_\mu(\tau)\vartheta_\mu(\tau,z),$$

where the functions $F_\mu(\tau)$ are uniquely determined by F and μ and given by

$$F_\mu(\tau) = \sum_{\substack{N \\ 4N - \mu^t\mu \le 0}} A(N,\mu) e\left(\frac{1}{4}tr\left(\left(4N - \mu^t\mu\right)\overline{\tau}\right)\right).$$

Next, we shortly review Siegel modular forms of half integral weight. We put $\theta_{m',m''}(\tau) = \theta_{m',m''}(\tau, 0)$. In order to define an automorphy factor of half integral weight, we put

$$\theta(\tau) \ = \ \theta_{0,0}(2\tau, 0) \ = \ \sum_{p \in \mathbb{Z}^n} e({}^t p \tau p) \ .$$

We denote by $\Gamma_0^{(n)}(4)$ the subgroup of $Sp(n, \mathbb{Z})$,

$$\Gamma_0^{(n)}(4) \ = \ \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(n, \mathbb{Z}) \ ; \ c \equiv 0 \, mod \, 4 \right\} \ .$$

We often drop the "(n)" and write $\Gamma_0(4)$ instead of $\Gamma_0^{(n)}(4)$ if no confusion is likely. We define a character ψ by $\psi(t) = \left(\frac{-4}{t}\right)$ for any odd integer t, where $\left(\frac{*}{*}\right)$ is the Legendre symbol. We consider a character of $\Gamma_0(4)$ defined by $\psi(\det d)$ for any $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$. By abuse of language, we denote this character also by ψ. It is well known that

$$\theta(M\tau)^2/\theta(\tau)^2 \ = \ \psi(M) \, \det(c\tau + d) \quad \text{ for any } \quad M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4).$$

By virtue of the above identity, we define by $\left(\frac{\theta(M\tau)}{\theta(\tau)}\right)^{2k-1}$ an automorphy factor of weight $k - 1/2$.

Definition 2 (Siegel modular forms of half integral weight). *Let $k \in \mathbb{Z}$, and χ be a character on $\Gamma_0(4)$. We say that a holomorphic function h on $\mathcal{H}_n$ is a Siegel modular form of weight $k - 1/2$ of degree n with character χ if h satisfies the following conditions (1), (2).*

(1) $\quad h(M\tau) = \chi(M) \left(\frac{\theta(M\tau)}{\theta(\tau)}\right)^{2k-1} h(\tau)$, *for any* $M \in \Gamma_0(4)$.

(2) *h is holomorphic at cusps (This condition is satisfied automatically when $n \geq 2$ by Koecher principle).*

Moreover if h satisfies the following condition (3), we say h is a cusp form.

(3) *The function $\det(Y)^{\frac{1}{2}(k-\frac{1}{2})}|h(\tau)|$ is bounded on $\mathcal{H}_n$, where Y is the imaginary part of τ.*

We denote by $M_{k-1/2}(\Gamma_0(4), \chi)$ (resp. $S_{k-1/2}(\Gamma_0(4), \chi)$) the space of Siegel modular forms (resp. Siegel cusp forms) of weight $k - 1/2$ with character χ.

Let l be an integer and let $h \in M_{k-1/2}(\Gamma_0(4), \psi^l)$, then the function h has the Fourier expansion $h(\tau) = \sum_T c(T) \, e(tr(T\tau))$, where

T runs over all symmetric half integral matrices. The above Fourier coefficients satisfy $c(T) = 0$, unless T is positive semi-definite. We define a subspace $M^+_{k-1/2}(\Gamma_0(4), \psi^l)$ of $M_{k-1/2}(\Gamma_0(4), \psi^l)$ by

$$\begin{aligned} & M^+_{k-1/2}(\Gamma_0(4), \psi^l) \\ = & \Big\{ h(\tau) \in M_{k-1/2}(\Gamma_0(4), \psi^l) \ ; \text{ the coefficients satisfy } \quad c(T) = 0 \ , \\ & \qquad \text{unless } T \equiv (-1)^{k+l+1} \mu\, {}^t\mu \ \bmod\ 4\, Sym^*(n, \mathbb{Z}) \text{ for some } \mu \in \mathbb{Z}^n \Big\} \ . \end{aligned}$$

We put

$$S^+_{k-1/2}(\Gamma_0(4), \psi^l) = M^+_{k-1/2}(\Gamma_0(4), \psi^l) \cap S_{k-1/2}(\Gamma_0(4), \psi^l) \ .$$

We say that $M^+_{k-1/2}(\Gamma_0(4), \psi^l)$ is the plus space. These are analogues of the "plus space" for general degree n with character ψ^l. This "plus space" was first defined for $n = 1$, $l = 0$ and $k \in \mathbb{Z}$ by Kohnen [8], and was generalized for $n > 1$, $l = 0$, and $k \in 2\mathbb{Z}$ by Ibukiyama [6], for $n > 1$, $l \equiv k \pmod 2$ by Hayashida-Ibukiyama [5].

Theorem 1. *Let k be a natural number. For $F(\tau) \in J^+_{k,1}$, we set $F(\tau) = \sum_{\mu \in (\mathbb{Z}/2\mathbb{Z})^n} F_\mu(\tau) \vartheta_\mu(\tau, z)$. We define a function $\sigma(F)(\tau)$ on $\mathcal{H}_n$ by :*

$$\sigma(F)(\tau) \ = \sum_{\mu \in (\mathbb{Z}/2\mathbb{Z})^n} F_\mu(-4\,\overline{\tau}) \ .$$

Then $\sigma(F)$ belongs to $M^+_{k-1/2}(\Gamma_0(4), \psi^{k-1})$. Moreover, the mapping $\sigma : F \rightarrow \sigma(F)$ induces the following linear isomorphisms over $\mathbb{C}$,

$$J^+_{k,1} \ \cong \ M^+_{k-1/2}(\Gamma_0(4), \psi^{k-1}) \ ,$$

and

$$J^{+\, cusp}_{k,1} \ \cong \ S^+_{k-1/2}(\Gamma_0(4), \psi^{k-1}) \ .$$

Remark 1. If degree n is odd and integer k is even, then $M_{k-1/2}(\Gamma_0(4), \psi) = \{0\}$ and $J^+_{k,1} = \{0\}$. We can show this fact by using an equality $F(\tau, -z) = -F(\tau, z)$ and by using an equality of Fourier coefficients $A(N, r) = A(N, -r)$.

Remark 2. We denote by $J^-_{k,1}$ the space of *holomorphic* Jacobi forms of weight k of index 1 of degree n (cf [4], [6]). Let $\epsilon = (-1)^{k+l+1}$, then we can write these linear isomorphisms together including holomorphic ones as follows;

$$M^{+}_{k-1/2}(\Gamma_0(4), \psi^l) \cong J^{\epsilon}_{k,1} \ .$$

2.2 Hecke operators.

In this subsection we review the action of Hecke operators on $J^{+}_{k,1}$ and $M^{+}_{k-1/2}(\Gamma_0(4), \psi^{k-1})$ and describe our results. First we describe Hecke operators for skew-holomorphic Jacobi forms of index 1. We define

$$\begin{aligned} & GSp^{+}(n, \mathbb{R}) \\ = \ & \left\{ M \in GL(2n, \mathbb{R}) \ ; \ M J_n {}^tM \ = \ \gamma(M)\, J_n \ , \text{ for some } \gamma(M) \in \mathbb{R}^{+} \right\} . \end{aligned}$$

For odd prime p and natural number δ, we define

$$V_n(p^{2\delta}) = \left\{ M \in GSp^{+}(n, \mathbb{R}) \cap M_{2n}(\mathbb{Z}) \ ; \ M J_n^t M \ = \ p^{2\delta} J_n \right\} .$$

Let F be a function on $\mathcal{H}_n \times \mathbb{C}^n$. We define an action of $V_n(p^{2\delta})$ as follows

$$\begin{aligned} & (F|_k M)(\tau, z) \\ = \ & p^{2kn\delta} e(-{}^t z (C\tau + D)^{-1} Cz) \overline{\det(C\tau + D)}^{1-k} |\det(C\tau + D)|^{-1} \\ & \times F\left((A\tau + B)(C\tau + D)^{-1}, p^{\delta t}(C\tau + D)^{-1} z \right) , \end{aligned}$$

where k is an integer and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in V_n(p^{2\delta})$.
We define an action of $M_{1,2n}(\mathbb{R})$ as follows,

$$(F|X)(\tau, z) \ = \ e({}^t\lambda\tau\lambda + 2{}^t\lambda z) \ F(\tau, z + \tau\lambda + \mu) \ ,$$

where $X = ({}^t\lambda, {}^t\mu) \in M_{1,2n}(\mathbb{R})$, and λ, $\mu \in M_{n,1}(\mathbb{R})$.

For integer s $(0 \le s \le n)$, we put $K_s = \begin{pmatrix} 1_{n-s} & & & \\ & p1_s & & \\ & & p^2 1_{n-s} & \\ & & & p1_s \end{pmatrix}$,
and we put a double coset $T_s(p^2) = Sp(n, \mathbb{Z}) K_s Sp(n, \mathbb{Z})$.

For any $F \in J_{k,1}^{+}$, we define Hecke operator $T_s(p^2)$ by

$$F|_k T_s(p^2) \quad = \sum_{M \in Sp(n,\mathbb{Z})\backslash T_s(p^2)} \; \sum_{X \in (\mathbb{Z}/p\mathbb{Z})^{2n}} (F|_k M)|X \ ,$$

where definition does not depend on the choice of the representatives in the summation, and we have $F|_k T_s(p^2) \in J_{k,1}^{+}$.

Next we review the Hecke theory of Siegel modular forms of half integral weight in Zhuravlev [12]. Let $\widetilde{GSp^+(n,\mathbb{R})}$ be the universal covering group of $GSp^+(n,\mathbb{R})$, namely $\widetilde{GSp^+(n,\mathbb{R})}$ consists of pairs of $M \in GSp^+(n,\mathbb{R})$ and a function $\phi(\tau)$ on $\mathcal{H}_n$ for which $|\phi(\tau)| = |\det M|^{-1/4} |\det(C\,\tau + D)|^{1/2}$ where $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in GSp^+(n,\mathbb{R})$. For (M_1, ϕ_1), $(M_2, \phi_2) \in \widetilde{GSp^+(n,\mathbb{R})}$, the product is defined by

$$(M_1, \phi_1(\tau)) \cdot (M_2, \phi_2(\tau)) \quad = \quad (M_1 M_2, \phi_1(M_2 \tau) \phi_2(\tau)).$$

We can embed $\Gamma_0(4)$ to $\widetilde{GSp^+(n,\mathbb{R})}$ by

$$\Gamma_0(4) \ni M \quad \to \quad (M, \theta(M\tau)\theta(\tau)^{-1}) \in \widetilde{GSp^+(n,\mathbb{R})}$$

and we denote by $\widetilde{\Gamma_0(4)}$ the image of $\Gamma_0(4)$. We put $\widetilde{K_s} = (K_s, p^{(n-s)/2}) \in \widetilde{GSp^+(n,\mathbb{R})}$. The $\widetilde{\Gamma_0(4)}$-double coset decomposition of $\widetilde{K_s}$ is known by Zhuravlev [11], and we put $\widetilde{T_s}(p^2) = \widetilde{\Gamma_0(4)}\widetilde{K_s}\widetilde{\Gamma_0(4)} = \bigcup_v \widetilde{\Gamma_0(4)}\widetilde{M_v}$. Let χ be a Dirichlet character modulo 4, we define an action of $\widetilde{T_s}(p^2)$ on $M_{k-1/2}(\Gamma_0(4), \chi)$ as follows :

$$F|_{k-1/2,\chi} \widetilde{T_s}(p^2) \quad = \quad \sum_v \chi(\det(D_v))^{-1} \phi_v(\tau)^{-2k+1} F(M_v \tau),$$

where $F \in M_{k-1/2}(\Gamma_0(4), \chi)$, we put $(M_v, \phi_v(\tau)) = \widetilde{M_v}$, $\begin{pmatrix} * & * \\ * & D_v \end{pmatrix} = M_v$.

Theorem 2. *The linear isomorphism of Theorem* 1 *commutes with Hecke operators. Namely, for every $F \in J_{k,1}^{+}$, for every odd prime p, and for every $T_s(p^2)$, we have*

$$\begin{aligned} &\left(\frac{-1}{p}\right)^{s(k-1)} p^{-2kn-s/2} \sigma(F|_k T_s(p^2)) \\ = \ & p^{-n(2k-1)/2} \sigma(F)|_{k-1/2,\psi^{k-1}} \widetilde{T_s}(p^2) \ . \end{aligned}$$

3 Proofs

In this section we shall prove Theorem 1 and Theorem 2.

3.1 Proof of Theorem 1.

Let $F(\tau, z) = \sum_{\mu \in (\mathbb{Z}/2\mathbb{Z})^n} F_\mu(\tau)\vartheta_\mu(\tau, z)$ be an element of $J^+_{k,1}$. First, we shall show that the holomorphic function $\sigma(F)(\tau) = \sum_{\mu \in (\mathbb{Z}/2\mathbb{Z})^n} F_\mu(-4\overline{\tau})$ belongs to $M^+_{k-\frac{1}{2}}(\Gamma_0(4), \psi^{k-1})$.

We denote by $GL(n, \mathbb{Z})$ the group of invertible matrices of size n with entries in $\mathbb{Z}$. In order to show that $\sigma(F)$ satisfies the condition (1) of the definition2, we need the following lemma proved by Ibukiyama [6].

Lemma 1.
The group $\Gamma_0(4)$ is generated by the following three kinds of elements :

$$v(4s) = \begin{pmatrix} 1_n & 0 \\ 4s & 1_n \end{pmatrix}, \quad u(s') = \begin{pmatrix} 1_n & s' \\ 0 & 1_n \end{pmatrix}, \quad \textit{and } t(a) = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix}.$$

where $s, s' \in Sym(n, \mathbb{Z})$, and $a \in GL(n, \mathbb{Z})$. ■

By straightforward calculation, we get $\sigma(F)(u(s')\tau) = \sigma(F)(\tau)$ and $\sigma(F)(t(a)\tau) = \psi(\det\ a)^{k-1}\sigma(F)(\tau)$.

Now, we shall show an equality $\sigma(F)(v(4s)\tau) = \left(\frac{\theta(v(4s)\tau)}{\theta(\tau)}\right)^{2k-1}\sigma(F)(\tau)$. We need the next proposition proved by Ibukiyama [6].

Proposition 1. *For $s \in Sym(n, \mathbb{Z})$ and $v(s) = \begin{pmatrix} 1_n & 0 \\ s & 1_n \end{pmatrix}$, we have*

$$\begin{aligned} & \vartheta_\mu(v(s)\tau, {}^t(s\tau + 1_n)^{-1}z) \\ = \ & 2^{-n}\det(s\tau + 1_n)e({}^t z(s\tau + 1_n)^{-1}sz)\,\theta(\frac{1}{4}\tau)\,\theta(\frac{1}{4}v(s)\tau)^{-1} \\ & \times \sum_{\nu,\kappa \in (\mathbb{Z}/2\mathbb{Z})^n} e(-\frac{1}{2}{}^t\nu\mu)e(-\frac{1}{4}{}^t\nu s\nu)e(\frac{1}{2}{}^t\kappa\nu)\vartheta_\kappa(\tau, z). \end{aligned}$$

■

By using this proposition and the condition (2) of the definition1, we obtain

$$
\begin{aligned}
&\overline{\det(s\tau+1_n)}^{k-1}|\det(s\tau+1_n)|\,e({}^tz(s\tau+1_n)^{-1}sz)\\
&\times\sum_{\mu\in(\mathbb{Z}/2\mathbb{Z})^n}F_\mu(\tau)\,\vartheta_\mu(\tau,z)\\
&=\sum_{\mu\in(\mathbb{Z}/2\mathbb{Z})^n}F_\mu(v(s)\tau)\,\vartheta_\mu(v(s)\tau,{}^t(s\tau+1_n)^{-1}z)\\
&=2^{-n}\,\det(s\tau+1_n)\,e({}^tz(s\tau+1_n)^{-1}sz)\,\theta(\frac{1}{4}\tau)\,\theta(\frac{1}{4}v(s)\tau)^{-1}\\
&\quad\times\sum_{\mu,\nu,\kappa\in(\mathbb{Z}/2\mathbb{Z})^n}F_\mu(v(s)\tau)\,e(-\frac{1}{2}{}^t\nu\mu)\,e(-\frac{1}{4}{}^t\nu s\nu)\,e(\frac{1}{2}{}^t\kappa\nu)\,\vartheta_\kappa(\tau,z)\ .
\end{aligned}
$$

Hence, by the uniqueness of F_μ, we have

$$
\begin{aligned}
&F_\kappa(\tau)\,\overline{\det(s\tau+1_n)}^{k-1}|\det(s\tau+1_n)|\,\det(s\tau+1_n)^{-1}\\
=\ &2^{-n}\,\theta(\frac{1}{4}\tau)\,\theta(\frac{1}{4}v(s)\tau)^{-1}\sum_{\mu,\nu\in(\mathbb{Z}/2\mathbb{Z})^n}F_\mu(v(s)\tau)\\
&\times e(-\frac{1}{2}{}^t\nu\mu)\,e(-\frac{1}{4}{}^t\nu s\nu)\,e(\frac{1}{2}{}^t\kappa\nu)\ .
\end{aligned}
$$

By easy calculation, we get $v(s)\cdot(-4\,\overline{\tau}) = -4\,\overline{v(-4s)\cdot\tau}$. If we substitute τ by $-4\,\overline{\tau}$, then we get

$$
\begin{aligned}
&(\sigma(F))(\tau)\,\overline{\det(-4s\overline{\tau}+1_n)}^{k-1}\,|\det(-4s\overline{\tau}+1_n)|\,\det(-4s\overline{\tau}+1_n)^{-1}\\
&=\ 2^{-n}\,\theta(-\overline{\tau})\,\theta(-\overline{v(-4s)\tau})^{-1}\\
&\quad\times\sum_{\kappa,\nu,\mu\in(\mathbb{Z}/2\mathbb{Z})^n}F_\mu(-4\overline{v(-4s)\tau})\,e(-\frac{1}{2}{}^t\nu\mu)\,e(-\frac{1}{4}{}^t\nu s\nu)\,e(\frac{1}{2}{}^t\kappa\nu)\\
&=\ \theta(-\overline{\tau})\,\theta(-\overline{v(-4s)\tau})^{-1}\,(\sigma(F))(v(-4s)\tau)\ .
\end{aligned}
$$

On the other hand, we get identities $\theta(-\overline{\tau}) = \overline{\theta(\tau)}$, $\det(-4s\overline{\tau}+1_n) = \left(\frac{\theta(v(4s)\cdot(-\overline{\tau}))}{\theta(-\overline{\tau})}\right)^2$, and $|\det(-4s\overline{\tau}+1_n)| = \left(\frac{\theta(v(-4s)\tau)}{\theta(\tau)}\right)\left(\frac{\theta(-\overline{v(-4s)\tau})}{\theta(-\overline{\tau})}\right)$ by straightforward calculation.
Hence, we get $\sigma(F)(v(-4s)\tau) = \left(\frac{\theta(v(-4s)\tau)}{\theta(\tau)}\right)^{2k-1}\sigma(F)(\tau)$.
This completes the proof of automorphy of $\sigma(F)$ for $v(4s)$.

Consequently, we have $\sigma(F)\in M^+_{k-1/2}(\Gamma_0(4),\psi^{k-1})$.

Next we shall show that the map σ is a bijection.
We assume that $h(\tau)$ is an element of $M^+_{k-1/2}(\Gamma_0(4),\psi^{k-1})$, then the function $h(\tau)$ has the Fourier expansion $h(\tau) \;=\; \sum_{M\geq 0} C(M)\, e(tr(M\tau))$.
For each $\mu \in (\mathbb{Z}/2\mathbb{Z})^n$, we define a function h_μ by

$$h_\mu(\tau) \;=\; \sum_{\substack{N \\ 4N+\mu^t\mu \geq 0}} C(4N+\mu^t\mu)\, e(\frac{1}{4}tr((4N+\mu^t\mu)\tau)).$$

Then we get $h(\tau) = \sum_{\mu\in(\mathbb{Z}/2\mathbb{Z})^n} h_\mu(4\tau)$. We put

$$G(\tau,z) \;=\; \sum_{\mu\in(\mathbb{Z}/2\mathbb{Z})^n} h_\mu(-\overline{\tau})\, \vartheta_\mu(\tau,z).$$

Our purpose is to show that the function $G(\tau,z)$ belongs to $J^+_{k,1}$.

Now, for any $x,y\in\mathbb{Z}^n$, the theta function ϑ_μ satisfy next equality,

$$\vartheta_\mu(\tau,z+\tau x+y) \;=\; e(-{}^t x\tau x - 2{}^t xz)\, \vartheta_\mu(\tau,z).$$

So $G(\tau,z)$ satisfies condition (1) of the definition 1. By definition of $G(\tau,z)$, we can show that $G(\tau,z)$ satisfies condition (3) of the definition 1.

In order to show that $G(\tau,z)$ satisfies condition (2) of the definition 1, we check the automorphy of $G(\tau,z)$ for three type generators of $Sp(n,\mathbb{Z})$.

By easy calculation, we get $G(u(s)(\tau,z)) = G(\tau,z)$ and $G(t(a)(\tau,z)) = \psi(\det\, a)^{k-1}G(\tau,z)$. We need the following proposition to show the automorphy of $G(\tau,z)$ for $v(s)$.

Proposition 2. *For any symmetric integral matrix s and any integer κ, we have*

$$\begin{aligned} & h_\kappa(\tau)\left(\frac{\theta(\frac{1}{4}v(s)\tau)}{\theta(\frac{1}{4}\tau)}\right)^{2k-1} \\ = \;& 2^{-n}\sum_{\nu,\mu\in(\mathbb{Z}/2\mathbb{Z})^n} e(-\frac{1}{2}{}^t\nu\mu)\, e(\frac{1}{4}{}^t\nu s\nu)\, e(\frac{1}{2}{}^t\kappa\nu)\, h_\mu(v(s)\tau)\;. \end{aligned}$$

Proof. First we claim the following relation:

$$h_\kappa(\tau) \quad = \quad 2^{-n} \sum_{s_1 \in \Delta} e(\frac{1}{2}{}^t\kappa s_1 \kappa)\, h(\frac{1}{4}(\tau + 2s_1)) \ .$$

where s_1 runs over the set Δ of all diagonal matrices such that each diagonal component is 0 or 1. Indeed, for $s_1 \in \Delta$, it is easy to see that

$$h(\frac{1}{4}(\tau + 2s_1)) \quad = \quad \sum_{\mu \in (\mathbb{Z}/2\mathbb{Z})^n} h_\mu(\tau + 2s_1) \ = \ \sum_{\mu \in (\mathbb{Z}/2\mathbb{Z})^n} e(-\frac{1}{2}{}^t\mu s_1 \mu)\, h_\mu(\tau) \ .$$

and

$$\begin{aligned} \sum_{s_1 \in \Delta} e(-\frac{1}{2}{}^t\kappa s_1 \kappa)\, e(-\frac{1}{2}\,{}^t\mu s_1 \mu) \quad &= \quad \sum_{s_1 \in \Delta} e(-\frac{1}{2}{}^t(\kappa - \mu) s_1 (\kappa - \mu)) \\ &= \quad \begin{cases} 2^n & \cdots \quad \text{if } \kappa = \mu \ , \\ 0 & \cdots \quad \text{otherwise} \ . \end{cases} \end{aligned}$$

Hence, we get the above relation. Now, for any $s_1 \in \Delta$, we put

$$\gamma_s(s_1) \ = \ \begin{pmatrix} 1_n + 2s_1 s & -s_1 s s_1 \\ 4s & 1_n - 2ss_1 \end{pmatrix} \ .$$

Then it is easy to see that $\gamma_s(s_1) \in \Gamma_0(4)$, and $(v(s)\tau + 2s_1)/4 = \gamma_s(s_1)((\tau + 2s_1)/4)$. Hence, we get the following relation,

$$h\left(\frac{v(s)\tau + 2s_1}{4}\right) \ = \ \det(s\tau + 1_n)^{k-1} \frac{\theta(\frac{1}{4}(v(s)\tau + 2s_1))}{\theta(\frac{1}{4}(\tau + 2s_1))} h\left(\frac{\tau + 2s_1}{4}\right) \ .$$

We put $\varepsilon = (s_1)_0$ which is the diagonal vector of s_1. Here we quote an equality from [6] p.120,

$$\theta\left(\frac{v(s)\tau + 2s_1}{4}\right) \theta\left(\frac{\tau + 2s_1}{4}\right)^{-1} \ = \ \theta\left(\frac{v(s)\tau}{4}\right) \theta\left(\frac{\tau}{4}\right)^{-1} e\left(-\frac{{}^t\varepsilon s \varepsilon}{4}\right) \ .$$

Hence, we get

$$
\begin{aligned}
& h_\kappa(\tau)\left(\frac{\theta(\frac{1}{4}v(s)\tau)}{\theta(\frac{1}{4}\tau)}\right)^{2k-1} \\
&= 2^{-n}\det(s\tau+1_n)^{k-1}\frac{\theta(\frac{1}{4}v(s)\tau)}{\theta(\frac{1}{4}\tau)}\sum_{s_1\in\Delta} e(\frac{1}{2}{}^t\kappa s_1\kappa)h(\frac{1}{4}(\tau+2s_1)) \\
&= 2^{-n}\sum_{s_1\in\Delta}\det(s\tau+1_n)^{k-1}\frac{\theta(\frac{1}{4}(v(s)\tau+2s_1))}{\theta(\frac{1}{4}(\tau+2s_1))} \\
&\quad\times\quad e(\frac{1}{4}{}^t\varepsilon s\varepsilon)e(\frac{1}{2}{}^t\kappa s_1\kappa)h(\frac{1}{4}(\tau+2s_1)) \\
&= 2^{-n}\sum_{s_1\in\Delta} e(\frac{1}{2}{}^t\kappa s_1\kappa)h(\frac{1}{4}(v(s)\tau+2s_1))e(\frac{1}{4}{}^t\varepsilon s\varepsilon) \\
&= 2^{-n}\sum_{s_1\in\Delta}\sum_{\mu\in(\mathbb{Z}/2\mathbb{Z})^n} e(\frac{1}{2}{}^t\kappa s_1\kappa)e(\frac{1}{4}{}^t\varepsilon s\varepsilon)e(-\frac{1}{2}{}^t\mu s_1\mu)h_\mu(v(s)\tau) \\
&= 2^{-n}\sum_{\varepsilon,\mu\in(\mathbb{Z}/2\mathbb{Z})^n} e(\frac{1}{2}{}^t\kappa\varepsilon-\frac{1}{2}{}^t\mu\varepsilon+\frac{1}{4}{}^t\varepsilon s\varepsilon)h_\mu(v(s)\tau)\ .
\end{aligned}
$$

Thus we have proved proposition 2. ∎

By using this proposition 2, we obtain

$$
\begin{aligned}
& \left(\frac{\theta(\frac{1}{4}v(s)(-\overline{\tau}))}{\theta(\frac{1}{4}(-\overline{\tau}))}\right)^{2k-1}\sum_{\kappa\in(\mathbb{Z}/2\mathbb{Z})^n} h_\kappa(-\overline{\tau})\vartheta_\kappa(\tau,z) \\
&= 2^{-n}\sum_{\mu\in(\mathbb{Z}/2\mathbb{Z})^n} h_\mu(v(s)(-\overline{\tau})) \\
&\quad\times\quad\left(\sum_{\varepsilon,\kappa\in(\mathbb{Z}/2\mathbb{Z})^n} e(-\frac{1}{2}{}^t\mu\varepsilon)e(-\frac{1}{4}{}^t\varepsilon(-s)\varepsilon)e(\frac{1}{2}{}^t\kappa\varepsilon)\vartheta_\kappa(\tau,z)\right) \\
&= \theta(\frac{1}{4}\tau)^{-1}\theta(\frac{1}{4}v(-s)\tau)\ \det(-s\tau+1_n)^{-1}e(-{}^tz(-s\tau+1_n)^{-1}(-s)z) \\
&\quad\times\sum_{\mu\in(\mathbb{Z}/2\mathbb{Z})^n} h_\mu(v(s)(-\overline{\tau}))\vartheta_\mu(v(-s)\cdot(\tau,z))\ .
\end{aligned}
$$

Moreover, we get

$$
\begin{aligned}
&\left(\frac{\theta(\frac{1}{4}v(s)(-\overline{\tau}))}{\theta(\frac{1}{4}(-\overline{\tau}))}\right)^{2k-1} \frac{\theta(\frac{1}{4}\tau)}{\theta(\frac{1}{4}v(-s)\tau)} \det(-s\tau+1_n) \\
&= \overline{\left(\frac{\theta(v(-4s)\cdot\frac{1}{4}\tau)}{\theta(\frac{1}{4}\tau)}\right)}^{2k-1} \frac{\theta(v(-4s)\cdot\frac{1}{4}\tau)}{\theta(\frac{1}{4}\tau)} \\
&= \overline{\det(-s\tau+1_n)}^{k-1} \,|\det(-s\tau+1_n)|,
\end{aligned}
$$

hence, we have

$$
\begin{aligned}
G(v(-s)\cdot(\tau,z)) &= \overline{\det(-s\tau+1_n)}^{k-1} |\det(-s\tau+1_n)| \\
&\quad \times e\left({}^t z(-s\tau+1_n)^{-1}(-s)z\right) G(\tau,z) .
\end{aligned}
$$

From the above identity, we deduce that the function G satisfies the automorphy for $v(s)$, and G is an element of $J^+_{k,1}$. By definition, it is clear that $\sigma(G)=h$, and hence σ is a bijection.

Now we prove that *cusp forms* of plus space and *cusp forms* of skew-holomorphic Jacobi forms are linearly isomorphic. For $f \in S^+_{k-1/2}(\Gamma_0(4),\psi^{k-1})$, it is easy to see that $\sigma^{-1}(f) \in J^{+\,cusp}_{k,1}$.
For $F \in J^{+\,cusp}_{k,1}$, we shall show that $\sigma(F)$ is an element of $S^+_{k-1/2}(\Gamma_0(4),\psi^{k-1})$.

The following lemma is well known and the proof will be omitted here.

Lemma 2. *we have*

$$
\vartheta_r(-\tau^{-1},\tau^{-1}z) = 2^{-n/2}\det(-i\tau)^{1/2}\, e({}^t z\tau^{-1}z) \sum_{\mu\in(\mathbb{Z}/2\mathbb{Z})^n} e(\frac{1}{2}{}^t r\mu)\vartheta_\mu(\tau,z).
$$

where $\det(-i\tau)^{1/2} = \left(\int_{x\in\mathbb{R}^n} e^{\pi i {}^t x\tau x} d\,x\right)^{-1}$.

We put $F(\tau,z)=\sum_r F_r(-\overline{\tau})\vartheta_r(\tau,z) \in J^+_{k,1}$, then we obtain the following equality by using lemma 2,

$$
F_r(-\tau^{-1}) = 2^{-n/2}\, i^{n(k-1)}\,(\det(-i\tau)^{1/2})^{(2k-1)} \sum_{\mu\in(\mathbb{Z}/2\mathbb{Z})^n} e(\frac{1}{2}{}^t r\mu)F_\mu(\tau).
$$

We already saw that

$$
F_r(\tau+s) = e(\frac{1}{4}{}^t rsr)\,F_r(\tau), \text{ and } F_r(a\tau{}^t a) = \psi(\det a)^{k-1}\,F_{{}^t ar}(\tau),
$$

for each $s \in Sym(n,\mathbb{Z})$ and each $a \in GL(n,\mathbb{Z})$.

The equalities above lead to the following lemma.

Lemma 3. *If we consider $(F_r(\tau))_r$ as a column vector valued function, then we have*

$$(F_r(M\tau))_r \quad = \quad U(M,\tau)\,(F_r(\tau))_r \ , \ \textit{for each } M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{Z}) \ ,$$

where $|\det(C\tau + D)|^{-k+1/2}\, U(M,\tau)$ *is some unitary matrix of size* n.

Now, we put $F(\tau,z) = \sum_r F_r(-\overline{\tau})\vartheta_r(\tau,z) \in J_{k,1}^{+\,cusp}$. If we fix a constant $\lambda > 0$ then the function $\det(Y)^{\frac{1}{2}(k-\frac{1}{2})}|F_r(\tau)|$ is bounded on the domain $\det(Y) > \lambda$.

There exists some constant $c_n > 0$ that depends only on n, with the property that for any element $\tau \in \mathcal{H}_n$ there exists some $M \in Sp(n,\mathbb{Z})$ satisfying $Im(M\cdot\tau) > c_n$ (cf. [1]). We put a scalar valued function $h(\tau) = \sum_r F_r(\tau)$. By using previous lemma and the above fact, it can be shown that the function $\det(Y)^{\frac{1}{2}(k-\frac{1}{2})}|h(\tau)|$ is bounded on $\mathcal{H}_n$.

Because $\sigma(F)(\tau) = h(4\tau)$, the function $\det(Y)^{\frac{1}{2}(k-\frac{1}{2})}|\sigma(F)(\tau)|$ is bounded on $\mathcal{H}_n$. This means that the function $\sigma(F)$ is a cusp form.

This completes the proof of Theorem 1.

3.2 proof of Theorem 2.

In this subsection we shall show Theorem 2.

For odd primes p, explicit formulas for the left $\widetilde{\Gamma_0(4)}$-coset decompositions of the double cosets were determined by Zhuravlev [11]. Let K_s be a matrix in subsection 2.2, we recall $K_s = \begin{pmatrix} 1_{n-s} & & & \\ & p1_s & & \\ & & p^2 1_{n-s} & \\ & & & p1_s \end{pmatrix}$.

We start with the following lemma.

Lemma 4. *The left $Sp(n,\mathbb{Z})$-coset decomposition of the double coset $Sp(n,\mathbb{Z})K_sSp(n,\mathbb{Z})$ is given by*

$$Sp(n,\mathbb{Z})K_sSp(n,\mathbb{Z}) \quad = \quad \coprod_{i,j}\ \coprod_{A,B_1,B_2,U} Sp(n,\mathbb{Z})M_{i,j}(A,B_1,B_2)P_U,$$

where the notations and the summations are given as follows. The first product is over all integers i, j such that $s \le i \le n$ and $0 \le j \le n-i$. In the second product the matrix A runs over a full set of representatives for the subset of classes in $Sym(i,\mathbb{Z})/p\,Sym(i,\mathbb{Z})$ having rank $i-s$, and B_1, B_2 run over a full set of representatives for $M_{i,j}(\mathbb{Z})/p\,M_{i,j}(\mathbb{Z})$ and $Sym(j,\mathbb{Z})/p^2\,Sym(j,\mathbb{Z})$, respectively. The matrix U runs over a complete set of representatives of $(SL(n,\mathbb{Z}) \cap D_{i,j}^{-1}SL(n,\mathbb{Z})D_{i,j})\backslash SL(n,\mathbb{Z})$. We use $M_{i,j}(A,B_1,B_2)$ for $M_{i,j}(A,B_1,B_2) = \begin{pmatrix} p^2D_{i,j}^{-1} & X \\ 0 & D_{i,j} \end{pmatrix}$, where $D_{i,j} = \begin{pmatrix} 1_{n-i-j} & 0 & 0 \\ 0 & p1_i & 0 \\ 0 & 0 & p^2 1_j \end{pmatrix}$ and $X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & A & p\,{}^tB_1 \\ 0 & B_1 & B_2 \end{pmatrix}$. Finally, we use $P_U = \begin{pmatrix} {}^tU^{-1} & 0 \\ 0 & U \end{pmatrix}$, where $U \in SL(n,\mathbb{Z})$.

Proof. See Andrianov-Zhuravlev [2]. ∎

We need the following lemma proved by Zhuravlev [11]

Lemma 5. *Left $\widetilde{\Gamma}_0(4)$-coset decomposition of $\widetilde{\Gamma}_0(4)\widetilde{K_s}\widetilde{\Gamma}_0(4)$ is given by*

$$\widetilde{\Gamma}_0(4)\widetilde{K_s}\widetilde{\Gamma}_0(4) = \coprod_{i,j} \coprod_{A,B_1,B_2,U} \widetilde{\Gamma}_0(4)\widetilde{M}_{i,j}(A,B_1,B_2,U) ,$$

where the notations and the summations are given as follows. The summations with respect to i, j, A, B_1, B_2, and U are taken over the same set mentioned in lemma 4. There exists a regular matrix A_1 with the property that ${}^tVAV \equiv \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}$ $(mod\,p)$ for some unimodular matrix V. We denote by γ the rank of A_1 over $\mathbb{Z}/p\mathbb{Z}$. (Note $\gamma = i-s$ by assumption of A). We put $\varepsilon_p = 1$ or $\varepsilon_p = \sqrt{-1}$ for $p \equiv 1 \pmod 4$ or $p \equiv 3 \pmod 4$, respectively. We put $\kappa(A) = 1$ or ${\varepsilon_p}^{-\gamma}\left(\frac{(-1)^\gamma \det A_1}{p}\right)$ for $\gamma = 0$ or $\gamma > 0$, respectively. Finally, we put $\widetilde{M}_{i,j}(A,B_1,B_2,U) = (M_{i,j}(A,B_1,B_2)P_U, \kappa(A)p^{(i+2j-n)/2})$, where $M_{i,j}(A,B_1,B_2)$ and P_U are the same notations as in lemma 4.

Proof. See Zhuravlev [11], [12]. ∎

We put $F(\tau,z)=\sum_{N,r} A(N,r)e(tr(N\tau-\frac{1}{2}(4N-r^tr)iY)+{}^trz)\in J_k^+$,
and put $\sigma(F)=\sum_{T\geq 0} C(T)\,e(tr(T\tau))$. Then, by definition, we get

$$A(N,r) \;=\; C(-4N+r^tr).$$

We have the following two lemmas.

Lemma 6. *We set*

$$\sigma(F)|_{k-1/2,\psi^{k-1}}\widetilde{T}_s(p^2) \;=\; \sum_{T\geq 0} C(T;s)e(tr(T\tau)),$$

then

$$\begin{aligned} C(T;s) \;&=\; \sum_{M_v} p^{(n-i-2j)(2k-1)/2}C(p^{-2}D_vT\,{}^tD_v) \\ &\qquad \times\; e(p^{-2}tr(T\,{}^tD_vB_v))\kappa(M_v)^{-2k+1}\psi(p^i)^{k-1}, \end{aligned}$$

where $C(*)=0$ unless $*$ is half integral matrix. In the above summation, M_v runs over a complete set of representatives of left $\Gamma_0(4)$-coset of $\Gamma_0(4)K_s\Gamma_0(4)$, and we set $M_v=\begin{pmatrix} A_v & B_v \\ 0 & D_v\end{pmatrix}$ (see lemma 5) .

Proof. This lemma is proved by straightforward calculation. ∎

Lemma 7. *We put*

$$(F|_kT_s(p^2))(\tau,z)=\sum_{4N-r^tr\leq 0} A(N,r;s)e(tr(N\tau-\frac{1}{2}(4N-r^tr)iY)+^t rz),$$

then the Fourier coefficients $A(N,r;s)$ are written by

$$\begin{aligned} A(N,r;s) \;&=\; p^{2kn}\sum_{\lambda\in(\mathbb{Z}/p\mathbb{Z})^n}\sum_{M_v} p^{-k(i+2j)}A(N_v(\lambda),r_v(\lambda)) \\ &\qquad \times\; e(tr(N_v(\lambda)B_vD_v^{-1})), \end{aligned}$$

where

$$\begin{aligned} N_v(\lambda) \;&=\; \frac{1}{p^2}\,D_v(N-\frac{1}{4}r^tr+\frac{1}{4}\,(r-2\lambda)^t(r-2\lambda))\,{}^tD_v, \\ r_v(\lambda) \;&=\; \frac{1}{p}\,D_v(r-2\lambda). \end{aligned}$$

and we regard $A(N_v(\lambda), r_v(\lambda)) = 0$ if $N_v(\lambda)$ is not half integral or $r_v(\lambda)$ is not an integer vector. In the above summation, M_v runs over a complete set of representatives of left $Sp(n,\mathbb{Z})$-coset of $Sp(n,\mathbb{Z})K_s Sp(n,\mathbb{Z})$, and we set $M_v = \begin{pmatrix} A_v & B_v \\ 0 & D_v \end{pmatrix}$ (see lemma 4).

Proof. This is proved by straightforward calculation, and details are omited here. ■

By direct calculation, we get the following identities

$$A(N_v(\lambda), r_v(\lambda)) = C(p^{-2} D_v(-4N + r\,{}^t r)\,{}^t D_v)$$

and

$$\begin{aligned} e(tr(N_v(\lambda) B_v D_v^{-1})) &= e\left(-\frac{1}{p^2}\, tr((-N + \frac{1}{4}\, r\,{}^t r)\,{}^t D_v B_v)\right) \\ &\quad \times e\left(\frac{1}{4p^2}\, {}^t(r - 2\lambda)\,{}^t D_v B_v (r - 2\lambda)\right). \end{aligned}$$

So, we get an equality

$$\begin{aligned} A(N, r; s) &= p^{2kn} \sum_v p^{-k(i+2j)} C(p^{-2} D_v(-4N + r\,{}^t r)\,{}^t D_v) \\ &\quad \times e\left(-\frac{1}{p^2}\, tr((-N + \frac{1}{4}\, r\,{}^t r)\,{}^t D_v B_v)\right) \\ &\quad \times \sum_{\lambda \in (\mathbb{Z}/p\mathbb{Z})^n} e\left(\frac{1}{4p^2}\, {}^t(r - 2\lambda)\,{}^t D_v B_v (r - 2\lambda)\right). \end{aligned}$$

We calculate the second sum of right hand side of the above equality. Here we replace B_v by $-B_v$, then

$$\begin{aligned}
&\sum_{\lambda\in(\mathbb{Z}/p\mathbb{Z})^n} e(-\frac{1}{4p^2}{}^t(r-2\lambda)^t D_v B_v (r-2\lambda)) \\
&= p^j \sum_{\lambda'\in(\mathbb{Z}/p\mathbb{Z})^i} e\left(-\frac{1}{p}{}^t\lambda' A\lambda'\right) \\
&= p^{i+j-\gamma(M_v)}(\varepsilon_p\sqrt{p})^{\gamma(M_v)}\left(\frac{\det(-A_1)}{p}\right) \\
&= p^{i+j-\frac{1}{2}\gamma(M_v)}\varepsilon_p^{\gamma(M_v)}\left(\frac{(-1)^{\gamma(M_v)}\det A_1}{p}\right) \\
&= p^{(s+i+2j)/2}\kappa(M_v)^{-2k+1}\varepsilon_p^{2(k-1)\gamma(M_v)} \\
&= p^{(s+i+2j)/2}\left(\frac{-1}{p}\right)^{-s(k-1)}\kappa(M_v)^{-2k+1}\psi(p^i)^{k-1}.
\end{aligned}$$

In the above calculation, we used a relation $\gamma(M_v) = s - i$.

Hence, we get the following identity

$$\psi(p)^{s(k-1)}p^{-2kn-s/2}A(N,r;s) \quad = \quad p^{-n(2k-1)/2}\,C(-4N+r^t r;s).$$

This completes the proof of Theorem 2.

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