FOURIER-JACOBI EXPANSION AND THE IKEDA LIFT

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Dedicated to Professor Tomoyoshi Ibukiyama on his sixtieth birthday

ABSTRACT. In this article, we consider a Fourier-Jacobi expansion of Siegel modular forms generated by the Ikeda lift. There are two purposes of this article: first, to give an expression of L-function of certain Siegel modular forms of *half-integral weight* of *odd degree*; and secondly, to give a relation among Fourier-Jacobi coefficients of Siegel modular forms generated by the Ikeda lift.

0. INTRODUCTION

0.1. **Preliminary.** On 2001 T.Ikeda [Ik 01] constructed a lifting from elliptic modular forms to Siegel modular forms of any even degree. The existence of such lifting has previously been conjectured by W. Duke-Ö. Imamoḡlu (cf. [B-K 00]) and by T. Ibukiyama, independently (cf. [Ik 01, Introduction]). Since Ikeda lift is a generalization of the Saito-Kurokawa lift for arbitrary even degree, it seems natural to investigate the Fourier-Jacobi coefficients of Siegel modular forms generated by the Ikeda lift.

The present article has two purposes: first, to express the L-functions of certain Siegel modular forms of *half-integral weight* of *odd degree* as products of Hecke L-functions of elliptic modular forms (cf. Theorem 0.1); second, to give certain relations among Fourier-Jacobi coefficients of Siegel modular forms obtained by the Ikeda lift (cf. Theorem 0.2).

0.2. **Main Theorems.** We explain our results more precisely. We put $\Gamma_m := \mathrm{Sp}_m(\mathbb{Z})$. We denote by $S_k(\Gamma_m)$ the space of Siegel cusp forms of weight k of degree m, and denote by $J_{k,1}^{cusp}(\Gamma_m^J)$ the space of Jacobi cusp forms of weight k of index 1 of degree m (cf. §3.1), and denote by $S_{k-1/2}^+(\Gamma_0^{(m)}(4))$ the space of Siegel cusp forms in the generalized plus space of weight $k-1/2$ of degree m (cf. §2.1).

Let k and n be positive integers such that $k+n$ is even. We define the following map $\Psi^{(2n-1)}$ from $S_{2k}(\Gamma_1)$ to $S_{k+n-1/2}^+(\Gamma_0^{(2n-1)}(4))$ through the composition of three maps:

$$\Psi^{(2n-1)} \quad : \quad S_{2k}(\Gamma_1) \to S_{k+n}(\Gamma_{2n}) \to J_{k+n,1}^{cusp}(\Gamma_{2n-1}^J) \to S_{k+n-\frac{1}{2}}^+(\Gamma_0^{(2n-1)}(4)).$$

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In the above composition the first map is the Ikeda lift, and the second map is the map from Siegel modular forms to Fourier-Jacobi coefficients of index 1, and the last map is given by the isomorphisms between the space of Jacobi forms of index 1 and the generalized plus space.

We will give some symbols to explain main results. Let $g \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform of elliptic modular forms of weight $2k$ and $g_1 \in S_{k+n}(\Gamma_{2n})$ the image of the Ikeda lift of g (cf. [Ik 01]). Let $\phi_r \in J^{cusp}_{k+n,r}(\Gamma^J_{2n-1})$ be the r-th Fourier-Jacobi coefficient of g_1, i.e. ϕ_r is obtained through the Fourier-Jacobi expansion of g_1:

$$g_1(Z) = \sum_{r>0} \phi_r(\tau, z) e(r\tau'), \tag{0.1}$$

where $Z = \left(\begin{smallmatrix} \tau & z \\ {}^t z & \tau' \end{smallmatrix}\right) \in \mathfrak{H}_{2n}$, $\tau \in \mathfrak{H}_{2n-1}$ and $\tau' \in \mathfrak{H}_1$. Then there exists a holomorphic function $\Psi^{(2n-1)}(g) \in S^+_{k+n-1/2}(\Gamma_0^{(2n-1)}(4))$ which corresponds to $\phi_1 \in J^{cusp}_{k+n,1}(\Gamma^J_{2n-1})$ through the isomorphism between two spaces $S^+_{k+n-1/2}(\Gamma_0^{(2n-1)}(4))$ and $J^{cusp}_{k+n,1}(\Gamma^J_{2n-1})$ (cf. [E-Z 85] for $n = 1$, [Ib 92] for $n > 1$, see also Theorem 3.2).

The following theorem is the first result of this article.

Theorem 0.1. *Let $g \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform. For an even integer $k+n$, let $\Psi^{(2n-1)}(g) \in S^+_{k+n-1/2}(\Gamma_0^{(2n-1)}(4))$ be the Siegel modular form constructed as above.*

Then, the form $\Psi^{(2n-1)}(g)$ is an eigenform for all Hecke operators acting on the generalized plus space. Moreover the L-function of $\Psi^{(2n-1)}(g)$ satisfies the following identity up to Euler 2-factor:

$$L(s, \Psi^{(2n-1)}(g)) \ = \ \prod_{i=1}^{2n-1} L(s + k + n - 1/2 - i, g).$$

Here $L(s, \Psi^{(2n-1)}(g))$ it the L-function of $\Psi^{(2n-1)}(g)$ introduced by V.G.Zhuravlev [Zh 84] (*see also* §*2.3*)*, and $L(s, g)$ is the Hecke L-function of g.*

We now explain our second result in this paper. Let g and ϕ_r be the same symbols stated before Theorem 0.1. For any prime p, we set parameters $\{\alpha_p^{\pm}\}$ of g through

$$\alpha_p + \alpha_p^{-1} = a(p) p^{-k+1/2}, \tag{0.2}$$

where $a(p)$ is the p-th Fourier coefficient of g. In this article we call the sequence of these parameters $\{\alpha_p^{\pm}\}_p$ *the Satake parameters* of g.

Then our second result is as follows:

Theorem 0.2. *For any positive integer r, we have the identity:*

$$\phi_r \ = \ \phi_1|_{k+n} D_{2n-1}(r, \{\alpha_p\}_p).$$

Here $D_{2n-1}(r,\{\alpha_p\}_p)$ is defined through the formal Dirichlet-series

$$\sum_{r>0} \frac{D_{2n-1}(r,\{\alpha_p\}_p)}{r^s} := \prod_{p:prime} (1 - G_p(\alpha_p)T^J(p)p^{\frac{1}{2}(n-1)(n+2)-s} + T^J_{0,2n-1}(p^2)p^{2n(2n-1)-1-2s})^{-1} ,$$

where we put a constant

$$G_p(\alpha_p) \;:=\; \prod_{1\le i\le n-1} \{(1+\alpha_p p^{\frac{1}{2}-i})(1+\alpha_p^{-1}p^{\frac{1}{2}-i})\}^{-1}$$

which depends on the Satake parameters $\{\alpha_p^{\pm}\}_p$ of g. (When $n=1$ we regard $G_p(\alpha_p)$ as 1). Here $T^J(p)$ (resp. $T^J_{0,2n-1}(p^2)$) is a Hecke operator which changes the index of Jacobi forms times p (resp. p^2). As for the definitions $|_{k+n}$, $T^J(p)$ and $T^J_{0,2n-1}(p^2)$, see §3.2 and §5.

When $n=1$, the operator $D_1(r,\{\alpha_p\}_p)$ does not depend on the choice of g and coincides with V_r-operator in [E-Z 85, p. 41]. *If we admit that g is the usual Eisenstein series of weight $2k$ with respect to $SL(2,\mathbb{Z})$, then $\{\alpha_p^{\pm}\}_p$ equal $\{p^{\pm(k-\frac{1}{2})}\}_p$ and the operator $D_{2n-1}(r,\{p^{k-\frac{1}{2}}\}_p)$ coincides with the operator $D_{2n-1}(r)$ introduced in* [Ya 86, p. 310].

0.3. **Some remarks.** The above two results are shown by the structure of Fourier coefficients of Siegel modular forms obtained by the Ikeda lift. Namely, the Fourier coefficients of such Siegel modular forms inherit some properties of Fourier coefficients of Siegel-Eisenstein series. On the other hand, Fourier coefficients or Fourier-Jacobi coefficients of Siegel-Eisenstein series have already been well studied.

Before we show Theorem 0.1, we will show that Jacobi-Eisenstein series of arbitrary index are Hecke eigenforms except for some local Hecke operators (cf. Proposition 4.1). Thus we obtain also the fact that Cohen-Eisenstein series of higher degree are eigenforms for any Hecke operator (cf. Corollary 4.2). Theorem 0.1 follows from this fact and an analogous theorem to Zharkovskaya's theorem for half-integral weight.

We remark that the identity in Theorem 0.2 can be regarded as a generalization of Maass relation, because it can be translated as a relation among Fourier coefficients of Siegel modular forms. However, it is different from the relation given in [Ko 02]. In [Ko 02], Fourier-Jacobi coefficients of *matrix index* of size $(2n-1)\times(2n-1)$ were treated, and in this paper we treat Fourier-Jacobi coefficients of *integer index*.

This article is organized as follows: in §1, we prepare some symbols and recall the construction of the Ikeda lift. In §2, we recall the definition of Siegel modular forms of half-integral weight and the L-function introduced by V.G.Zhuravlev. In §3, we describe the definition of Jacobi forms, and we review Hecke operators which act on the space of Jacobi forms. In §4, we describe the definition of Eisenstein series in spaces of

Siegel modular forms and of Jacobi forms. Furthermore, we show the fact that Jacobi-Eisenstein series is an eigenform for some Hecke operators. In §5, we describe a formula given by T.Yamazaki for Fourier-Jacobi coefficients of Siegel-Eisenstein series. In the final section, we give proofs of Theorem 0.1 and 0.2.

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Notation:

Throughout the present article, we use the following notation. For matrix $A \in \mathrm{M}_n(\mathbb{R})$, we set $e(A) := \exp(2\pi i \ \mathrm{tr}(A))$, where $\mathrm{tr}(A)$ is the trace of A. We denote by ${}^tA$ the transpose of a matrix A. We denote by $\mathfrak{N}_n$ (resp. $\mathfrak{N}_n^+$) the set of positive semi-definite (resp. positive definite) half-integral symmetric matrices of size n. We denote by $\mathfrak{H}_n$ the Siegel upper half space of size n, and denote by $\mathrm{Sp}_n(K)$ the symplectic group of matrix size $2n$ whose entries are in a commutative ring K. In particular, we put $\Gamma_n := \mathrm{Sp}_n(\mathbb{Z})$. We denote by $M_k(\Gamma_n)$ (resp. $S_k(\Gamma_n)$) the space of Siegel modular forms (resp. Siegel cusp forms) of weight k and of degree n with respect to Γ_n. Notations $|_k$ and $M \cdot \tau$ are slash operator and the action of $M \in \mathrm{GSp}_n^+(\mathbb{R})$ on $\tau \in \mathfrak{H}_n$ in usual sense.

We denote by $S[T] := {}^tTST$ *the Siegel's symbol* for matrices S, T. We put

$$\mathrm{GSp}_n^+(\mathbb{R}) := \{M \in \mathrm{GL}(2n,\mathbb{R}) \mid J_n[M] = \nu(M)J_n \ \text{ for some real number } \nu(M) > 0\},$$

where $J_n := \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$, and where 0_n (resp. 1_n) is the zero matrix (resp. the identity matrix) of size n, and where $\nu(M)$ is called *the similitude* of M. For positive integer r, we set

$$\mathfrak{N}_{2n-1,r} := \left\{ \left(\begin{smallmatrix} N & R \\ {}^tR & r \end{smallmatrix}\right) \mid N \in \mathfrak{N}_{2n-1}\,,\ R \in \mathbb{Z}^{2n-1}\,,\ 4N - r^{-1}R\,{}^tR \geq 0 \right\} \subset \mathfrak{N}_{2n},$$
$$\mathfrak{N}^+_{2n-1,r} := \left\{ \left(\begin{smallmatrix} N & R \\ {}^tR & r \end{smallmatrix}\right) \mid N \in \mathfrak{N}^+_{2n-1}\,,\ R \in \mathbb{Z}^{2n-1}\,,\ 4N - r^{-1}R\,{}^tR > 0 \right\} \subset \mathfrak{N}^+_{2n}.$$

1. The Ikeda lift

For a prime p and a positive definite half-integral symmetric matrix B, we denote by $\tilde{F}_p(B;X)$ a certain (reciprocal) Laurent polynomial which is related to a Siegel series. As for the explicit expression of $\tilde{F}_p(B;X)$ see [Ik 01, p. 645]. It is known that this Laurent polynomial $\tilde{F}_p(B;X)$ is reciprocal, namely $\tilde{F}_p(B;X)$ satisfies the identity $\tilde{F}_p(B;X) = \tilde{F}_p(B;X^{-1})$.

Let k and n be positive integers which satisfy $k+n \equiv 0 \pmod 2$. Let $g \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform. For prime p, we denote by $\{\alpha_p^{\pm}\}_p$ the Satake parameters of g (cf. the identity (0.2) in the previous section). Let $\Gamma_0^{(1)}(4)$ be the congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ consisting of the matrices whose left-lower part is divisible by 4. We denote by $h(z) = \sum_{m>0} c(m)\, e(mz) \in S^+_{k+\frac{1}{2}}(\Gamma_0^{(1)}(4))$ the elliptic modular form of weight $k+\frac{1}{2}$ in Kohnen plus space which corresponds to g by means of the Shimura correspondence (cf. [Sh 73], [Ko 80]).

For each positive-definite half-integral symmetric matrix B of size $2n$, we set $D_B := (-1)^n \det(2B)$, and $f_B := \sqrt{|D_B|\delta_B^{-1}}$, where δ_B is the absolute value of the discriminant of the quadratic field $\mathbb{Q}(\sqrt{D_B})/\mathbb{Q}$.

The following theorem is first conjectured by W. Duke-Ö. Imamoḡlu (cf. [B-K 00]) and solved by T. Ikeda [Ik 01].

Theorem 1.1 ([Ik 01])**.** *Let the notations be as above. We set*

$$A(B) := c(\delta_B)\, f_B^{k-\frac{1}{2}} \prod_{p|f_B} \tilde{F}_p(B;\alpha_p)$$

and $F(Z) := \sum_{B>0} A(B)\, e(BZ)$ *for* $Z \in \mathfrak{H}_{2n}$.

Then F belongs to $S_{k+n}(\Gamma_{2n}(\mathbb{Z}))$. Moreover F is a Hecke eigenform whose standard L-function satisfies the following identity:

$$L(s,F) = \zeta(s) \prod_{i=1}^{2n} L(s+k+n-i, g)\,.$$

where $L(s,g)$ is the usual Hecke L-function of g.

2. Siegel modular forms of half-integral weight

In this section, we recall the definition of the Siegel modular forms of half-integral weight, and shortly review Hecke theory of Siegel modular forms of half-integral weight introduced by V. G. Zhuravlev [Zh 83], [Zh 84].

2.1. Double covering group and Siegel modular forms of half-integral weight.

Let m be a positive integer. The double covering group $\widetilde{G}$ for $\mathrm{GSp}_m^+(\mathbb{R})$ consists of pairs (M,φ), where $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is in $\mathrm{GSp}_m^+(\mathbb{R})$, and where φ is a holomorphic function on $\mathfrak{H}_m$ which satisfies $|\varphi(\tau)|^2 = \det M^{-1/2} |\det(C\tau + D)|$. The group $\widetilde{G}$ has the group operation $(M,\varphi(\tau)) \cdot (L,\psi(\tau)) := (ML, \varphi(L\cdot\tau)\psi(\tau))$, for $(M,\varphi(\tau)),\ (L,\psi(\tau)) \in \widetilde{G}$.

Let $\theta^{(m)}(\tau) := \sum_{p\in\mathbb{Z}^m} e(\tau[p])$ $(\tau \in \mathfrak{H}_m)$ be the theta series. we put

$$\hat{\Gamma}_0^{(m)}(4) \;:=\; \left\{ \left(M, \frac{\theta^{(m)}(M\cdot\tau)}{\theta^{(m)}(\tau)} \right) \in \widetilde{G} \mid M \in \Gamma_0^{(m)}(4) \right\},$$

where $\Gamma_0^{(m)}(4) := \left\{ \begin{pmatrix} A & B \\ 4C & D \end{pmatrix} \in \Gamma_m \middle| A,B,C,D \in \mathrm{M}_m(\mathbb{Z}) \right\}$.

Let k be an integer. For any function F on $\mathfrak{H}_m$ and for $\xi = (M,\varphi) \in \widetilde{G}$, we set $(F|_{k-1/2}\xi)(\tau) := \varphi(\tau)^{-2k+1}F(M\cdot\tau)$.

A holomorphic function F on $\mathfrak{H}_m$ is said to be a *Siegel modular form of weight $k-1/2$ of degree m*, if F satisfies $F|_{k-1/2}\xi = F$ for any $\xi \in \hat{\Gamma}_0^{(m)}(4)$, moreover, when $m=1$ we require the condition that the function F is holomorphic at all cusps of $\Gamma_0^{(1)}(4)$. We denote by $M_{k-1/2}(\Gamma_0^{(m)}(4))$ the $\mathbb{C}$-vector space of Siegel modular forms of weight $k-1/2$ of degree m. We say F is a cusp form if F^2 is a cusp form of Siegel modular form of integral weight. We denote by $S_{k-1/2}(\Gamma_0^{(m)}(4))$ the space of cusp forms of Siegel modular forms of weight $k-1/2$ of degree m.

2.2. **Hecke operators on Siegel modular forms of half-integral weight.** For odd prime p, let $\hat{L}_p^m$ be the Hecke ring generated by double cosets $\hat{\Gamma}_0^{(m)}(4)\hat{k}_{s,p}\hat{\Gamma}_0^{(m)}(4)$ $(s = 0,...,m-1)$ and $\hat{\Gamma}_0^{(m)}(4)\hat{k}_{m,p}^{\pm}\hat{\Gamma}_0^{(m)}(4)$, where $\hat{k}_{s,p} := (k'_{s,p}, p^{(m-s)/2}) \in \widetilde{G}$ and where we put $k'_{s,p} := \mathrm{diag}\,(1_{m-s},\, p1_s,\, p^2 1_{m-s},\, p1_s)$ for $s = 0,...,m$.

For $F \in M_{k-1/2}(\Gamma_0^{(m)}(4))$ and for $H = \sum_v a_v \hat{\Gamma}_0^{(m)}(4)\hat{M}_v \in \hat{L}_p^m$ with $a_v \in \mathbb{C}$, we define $F|_{k-1/2}H := \sum_v a_v F|_{k-1/2}\hat{M}_v$. We denote by $\hat{T}_s(p^2)$ the Hecke operator which corresponds to $\hat{\Gamma}_0^{(m)}(4)\hat{k}_{s,p}\hat{\Gamma}_0^{(m)}(4)$. As for more explicit explanation about the Hecke ring, see [Zh 83], [Zh 84].

2.3. **Zeta functions and an analogous of Zharkovskaya's theorem.** Let $F \in M_{k-\frac{1}{2}}(\Gamma_0^{(n)}(4))$ be a Hecke eigenform, namely F is an eigenform for all operators $T \in \hat{L}_p^m$ and for all odd prime p. Then there exist so called p-parameters $\alpha_{i,p}^{\pm}$ $(i = 1,2,...,n)$ of F (cf. [Zh 84]). For sufficiently large $\mathrm{Re}(s)$, we define the L-function of F by

$$L(s,F) \;:=\; \prod_{\substack{p \\ p\neq 2}} \prod_{i=0}^{n} \left\{ (1-\alpha_{i,p}p^{-s})(1-\alpha_{i,p}^{-1}p^{-s}) \right\}^{-1}.$$

Let Φ be the Siegel-phi operator. Then the following theorem is known.

Theorem 2.1 ([O-K-K 89], [Ha 03]). *We assume $\Phi(F) \neq 0$. Then we have*

(1) $\Phi(F) \in M_{k-\frac{1}{2}}(\Gamma_0^{(n-1)}(4))$ *is a Hecke eigenform of weight $k-1/2$ of degree $n-1$,*

(2) *the L-function of F has the following decomposition,*

$$L(s,F) \;=\; L_1(s+k-n-1/2, E^{(1)}_{2k-2n})L(s,\Phi(F)),$$

where $L_1(s,E^{(1)}_{2k-2n}) := \prod_{p\neq 2}(1-p^{-s})^{-1}(1-p^{2k-2n-1-s})^{-1}.$

The claim (1) of the above theorem is a special case of [O-K-K 89, Theorem 5.1, 5.2], and the claim (2) is a special case of [Ha 03, Theorem 2].

3. Jacobi forms of higher degree

In this section, we review the Hecke operators acting on the space of Jacobi forms and review the isomorphism between the space of Jacobi forms of index 1 and a generalized plus space.

3.1. **The Jacobi group and Jacobi forms.** For positive integer m, we define

$$G^J_m \;:=\; \left\{ \left(\begin{smallmatrix} A & 0 & B & * \\ * & a & * & * \\ C & 0 & D & * \\ 0 & 0 & 0 & d \end{smallmatrix}\right) \in \mathrm{GSp}^+_{m+1}(\mathbb{R}) \;\middle|\; \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GSp}^+_m(\mathbb{R}),\ a,d\in\mathbb{R}^\times \right\}.$$

For $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{GSp}^+_m(\mathbb{R})$, $\lambda,\mu\in\mathbb{R}^m$, $d\in\mathbb{R}^\times$, $u\in\mathbb{R}$, we write

$$\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, d\right) := \begin{pmatrix} A & 0 & B & 0 \\ 0 & \nu d^{-1} & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \text{ and } ([\lambda,\mu,\kappa]) := \begin{pmatrix} 1 & 0 & 0 & \mu \\ {}^t\lambda & 1 & {}^t\mu & \kappa \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where ν is the similitude of $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, and where λ and μ are column vectors. We note that every element $M\in G^J_m$ has an expression $M = \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, d\right)([\lambda,\mu,\kappa])$.

By direct calculation we have

$$\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, d\right)^{-1}([\lambda,\mu,\kappa])\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, d\right) = \left(\left[\frac{d}{\nu}\lambda', \frac{d}{\nu}\mu', \frac{d^2}{\nu}\kappa\right]\right), \tag{3.1}$$

where $\begin{pmatrix} \lambda' \\ \mu' \end{pmatrix} = {}^t\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} \lambda \\ \mu \end{pmatrix}$, and where ν is the similitude of $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

We define an action of the group G^J_m on $\mathfrak{H}_m\times\mathbb{C}^m$ by

$$M\cdot(\tau,z) := \left((A\tau+B)(C\tau+D)^{-1}, \frac{\nu}{d}{}^t(C\tau+D)^{-1}(z+\tau\lambda+\mu)\right),$$

for $M = (\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right), d)([\lambda,\mu,u]) \in G^J_m$, $(\tau,z)\in\mathfrak{H}_m\times\mathbb{C}^m$, and where ν is the similitude of $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$.

For a positive integer r, we regard $e(r\tau')$ as a function on $\mathfrak{H}_{m+1}$, where $\left(\begin{smallmatrix} \tau & z \\ {}^t z & \tau' \end{smallmatrix}\right) \in \mathfrak{H}_{m+1}$, $\tau \in \mathfrak{H}_m$ and $\tau' \in \mathfrak{H}_1$. For an integer k we define a factor of automorphy

$$\begin{aligned} J_{k,r}(M,(\tau,z)) &:= (e(r\tau')|_k M)^{-1} e(r\nu d^{-2}\tau') \\ &= d^k \det(C\tau+D)^k \, e(r\nu d^{-2}(((C\tau+D)^{-1}C)[(z+\tau\lambda+\mu)])) \\ &\quad \times e(-r({}^t\lambda\tau\lambda + 2\,{}^t z\lambda + {}^t\mu\lambda + u)) \ , \end{aligned}$$

where $(\tau,z) \in \mathfrak{H}_m \times \mathbb{C}^m$, $\tau' \in \mathfrak{H}_1$ and $M = ((\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right), d)([\lambda,\mu,u]) \in G_m^J$. We note that the above $J_{k,r}(M,(\tau,z))$ does not depend on the choice of τ'. Moreover, for a function $\phi(\tau,z)$ on the space $\mathfrak{H}_m \times \mathbb{C}^m$, we define the slash operator $|_{k,r}$ by

$$(\phi|_{k,r}M)(\tau,z) := J_{k,r}(M,(\tau,z))^{-1}\phi(M\cdot(\tau,z)) \text{ for any } M \in G_m^J \ .$$

We define a discrete subgroup Γ_m^J of G_m^J by $\Gamma_m^J := \left\{M \in G_m^J \cap \Gamma_{m+1}\right\}$.

Let $m > 1$ be an integer. A holomorphic function ϕ on $\mathfrak{H}_m \times \mathbb{C}^m$ is said to be a *Jacobi form of index r of weight k of degree m*, if ϕ satisfies the identity $\phi|_{k,r}M = \phi$ for any $M \in \Gamma_m^J$. We note that a holomorphic function ϕ on $\mathfrak{H}_m \times \mathbb{C}^m$ is a Jacobi form, if and only if the holomorphic function $\hat{\phi}(\left(\begin{smallmatrix} \tau & z \\ {}^t z & \tau' \end{smallmatrix}\right)) := \phi(\tau,z)e(r\tau')$ ($\left(\begin{smallmatrix} \tau & z \\ {}^t z & \tau' \end{smallmatrix}\right) \in \mathfrak{H}_{m+r}$) satisfies $\hat{\phi}|_k M = \hat{\phi}$ for any $M \in \Gamma_m^J$.

Each Jacobi form $\phi(\tau,z)$ of degree m has the Fourier expansion:

$$\phi(\tau,z) = \sum_{\substack{N \in \mathfrak{N}_m, R \in \mathbb{Z}^m \\ 4N - r^{-1}R\,{}^tR \geq 0}} A(N,R)\, e(N\tau)\, e({}^tRz).$$

In the above expansion, if each Fourier coefficient $A(N,R)$ of ϕ vanishes unless $4N - r^{-1}R\,{}^tR > 0$, then we say ϕ is a *Jacobi cusp form*. In this article, we call $A(N,R)$ *the (N,R)-th Fourier coefficient of ϕ*.

We denote by $J_{k,r}(\Gamma_m^J)$ (resp. $J_{k,r}^{cusp}(\Gamma_m^J)$) the $\mathbb{C}$-vector space consisting of all Jacobi forms (resp. Jacobi cusp forms) of weight k, of index r and of degree m.

3.2. **Hecke operators acting on the space of Jacobi forms.** In this subsection, we introduce Hecke operators for Jacobi forms. (cf. [E-Z 85], [S-Z 89] for $m = 1$, and [Ya 86], [Gr 84], [Mu 89], [Ar 94] for $m > 1$.)

Let α be an element of $G_m^J \cap \mathrm{GSp}_{m+1}^+(\mathbb{Q})$. For the double coset $\Gamma_m^J\alpha\Gamma_m^J$, we define

$$\phi|\Gamma_m^J\alpha\Gamma_m^J \ := \ \nu(\alpha)^{\frac{m+1}{2}k - \frac{m(m+1)}{2}} \sum_{v=1}^{h} \phi|_{k,r}M_v,$$

where $\Gamma_m^J\alpha\Gamma_m^J = \displaystyle\bigcup_{v=1}^{h}\Gamma_m^J M_v$, $\phi \in J_{k,r}(\Gamma_m^J)$, and where $\nu(\alpha)$ is the similitude of α. Since Γ_m^J is written as a semi-direct product of $\mathrm{Sp}_n(\mathbb{Z})$ and certain Heisenberg group, it is not difficult to show that $h < \infty$ (see also the proof of [S-Z 89, Proposition 1.1].)

Let $\alpha = (((\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right), d), (\lambda,\mu,u)) \in G_m^J$ with similitude ν. For $\phi \in J_{k,r}(\Gamma_m^J)$ we obtain $\phi|\Gamma_m^J\alpha\Gamma_m^J \in J_{k,\nu d^{-2}r}(\Gamma_m^J)$. Moreover, if ϕ is a Jacobi cusp form, then $\phi|\Gamma_m^J\alpha\Gamma_m^J$ is also a Jacobi cusp form. We note that if $\nu d^{-2} r \notin \mathbb{N}$ then $\phi|\Gamma_m^J \alpha \Gamma_m^J = 0$, since $\phi|\Gamma_m^J \alpha \Gamma_m^J = (\phi|\Gamma_m^J \alpha \Gamma_m^J)|_{k,\nu d^{-2} r}([0,0,u]) = e(-\nu d^{-2} r u)\phi|\Gamma_m^J \alpha \Gamma_m^J$.

We put $k_{s,p}$:=diag $(1_{m-s},\, p1_s,\, p,\, p^2 1_{m-s},\, p1_s,\, p) \in G_m^J$. We recall that $k'_{s,p}$ was defined in § 2.2 as $k'_{s,p} = \mathrm{diag}\,(1_{m-s},\, p1_s,\, p^2 1_{m-s},\, p1_s)$. We have the following lemma.

Lemma 3.1. *The double coset $\Gamma_m^J k_{s,p} \Gamma_m^J$ is written as a union:*

$$\Gamma_m^J k_{s,p} \Gamma_m^J = \bigcup_{\substack{\lambda,\mu \in (\mathbb{Z}/p\mathbb{Z})^m \\ M \in \Gamma_m \backslash \Gamma_m\, k'_{s,p}\, \Gamma_m}} \Gamma_m^J \left(\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix} \right), p \right) \left(\left[\lambda, \mu, {}^t\lambda\mu \right] \right), \tag{3.2}$$

where $M = \left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix} \right)$ runs over a complete set of representatives of $\Gamma_m \backslash \Gamma_m\, k'_{s,p}\, \Gamma_m$. Moreover, on the right hand side the same Γ_m^J-right cosets appear p^{m+s} times.

Proof. It is not difficult to see that the both side of (3.2) coincides as sets.

The following action of $\gamma \in \Gamma_m^J$ on the cosets of the right hand side in (3.2) is transitive:

$$\Gamma_m^J \left(\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix} \right), p \right) \left(\left[\lambda, \mu, {}^t\lambda\mu \right] \right) \to \Gamma_m^J \left(\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix} \right), p \right) \left(\left[\lambda, \mu, {}^t\lambda\mu \right] \right) \gamma.$$

Hence each Γ_m^J-right coset appears equally many times in the right hand side of the identity (3.2). In particular, the coset $\Gamma_m^J k_{s,p}$ appears p^{m+s} times. This fact follows from a straightforward calculation by using the identity (3.1). Hence the same Γ_m^J-right cosets appear p^{m+s} times. □

3.3. **Isomorphism between the space of Jacobi forms of index 1 and a generalized plus space.** A generalized plus space is a certain subspace of the space of Siegel modular forms of half-integral weight. The plus space was first given in the case of degree $m = 1$ by W. Kohnen [Ko 80], and was generalized for general degree m by T. Ibukiyama [Ib 92]. Let k be an integer. We put a generalized plus space

$$M^+_{k-\frac{1}{2}}(\Gamma_0^{(m)}(4)) := \left\{ f \in M_{k-\frac{1}{2}}(\Gamma_0^{(m)}(4)) \,\middle|\, C(N) = 0 \text{ unless } N \in \mathbb{L}_{m,k} \right\},$$

where $C(N)$ denotes the N-th Fourier coefficient of f, and $\mathbb{L}_{m,k}$ is defined by

$$\mathbb{L}_{m,k} := \left\{ N \in \mathfrak{N}_m \mid N + (-1)^k \lambda {}^t\lambda \in 4\,\mathfrak{N}_m \text{ for some } \lambda \in \mathbb{Z}^m \right\}.$$

The following theorem has been known.

Theorem 3.2 ([E-Z 85], [Ib 92]). *For even integer k, the space of Jacobi forms of weight k of index 1 is linearly isomorphic to the generalized plus space of weight $k - 1/2$ of degree m. Namely we have the linear isomorphism*

$$\sigma : \; J_{k,1}(\Gamma_m^J) \cong M^+_{k-1/2}(\Gamma_0^{(m)}(4)).$$

Here the map σ is given as follows: let $\phi(\tau, z) = \sum_{N,R} A(N,R) e(N\tau) e({}^tRz) \in J_{k,1}(\Gamma_m^J)$, then the isomorphism σ is given by $\sigma(\phi)(\tau) = \sum_{M \in \mathbb{L}_{m,0}} A(N,R) e(M\tau)$, where $M = 4N - R{}^tR$, $N \in \mathfrak{N}_m$, and $R \in \mathbb{Z}^m$.

In particular, the isomorphism σ preserves the both spaces of cusp forms. Moreover, these are isomorphisms as Hecke algebra module, i.e. for $\phi \in J_{k,1}(\Gamma_m^J)$ we have

$$\phi|\Gamma_m^J k_{s,p} \Gamma_m^J \;=\; p^{-m^2-\frac{1}{2}m-\frac{1}{2}s}\,\sigma(\phi)|\hat{T}_s(p^2)\ .$$

4. The Jacobi-Eisenstein series and the Cohen-Eisenstein series of general degree

Let $k > m+2$ be an even integer and $E_k^{(m+1)}(Z)$ the Siegel-Eisenstein series of weight k and of degree $m+1$, i.e. we define

$$E_k^{(m+1)}(Z) \;:=\; \sum_{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_{0,m+1}\backslash \Gamma_{m+1}} \det(CZ+D)^{-k},$$

where

$$\begin{aligned} \Gamma_{0,m+1} \;&:=\; \{M \in \Gamma_{m+1} \mid 1|_k M = 1\} \\ &=\; \left\{ \left(\begin{smallmatrix} A & B \\ 0 & {}^tA^{-1} \end{smallmatrix}\right) \in \Gamma_{m+1} \mid A \in GL_{m+1}(\mathbb{Z})\ ,\ B \in M_{m+1}(\mathbb{Z})\ ,\ A^tB = B^tA \right\}\ . \end{aligned}$$

The Jacobi-Eisenstein series of degree 1 is introduced in [E-Z 85], and it is generalized for higher degree in [Zi 89].

To describe the definition of the Jacobi-Eisenstein series, we set a subgroup of Γ_m^J:

$$\begin{aligned} \Gamma_{0,m}^J \;&:=\; \left\{ M \in \Gamma_m^J \mid 1|_{k,1} M = 1 \right\} \\ &=\; \left\{ \left(\left(\begin{smallmatrix} A & B \\ 0 & {}^tA^{-1} \end{smallmatrix}\right), 1\right)([0,\mu,\kappa]) \in \Gamma_m^J \;\middle|\; \left(\begin{smallmatrix} A & B \\ 0 & {}^tA^{-1} \end{smallmatrix}\right) \in \Gamma_{0,m},\ \mu \in \mathbb{Z}^m,\ \kappa \in \mathbb{Z} \right\}. \end{aligned}$$

We denote by $E_{k,r}^{(m)}$ the Jacobi-Eisenstein series of index r which is defined by

$$E_{k,r}^{(m)}(\tau,z) \;:=\; \sum_{\gamma \in \Gamma_{0,m}^J \backslash \Gamma_m^J} (1|_{k,r}\gamma)(\tau,z)\ ,\quad \text{for } (\tau,z) \in \mathfrak{H}_m \times \mathbb{C}^m. \tag{4.1}$$

It is known by [Zi 89, Theorem 2.1] that if $k > m+2$, then the above sum converges and the form $E_{k,r}^{(m)}$ belongs to $J_{k,r}(\Gamma_m^J)$.

We denote by $E_{k-1/2}^{(m)} \in M_{k-1/2}^+(\Gamma_0^{(m)}(4))$ the form which corresponds to $E_{k,1}^{(m)} \in J_{k,1}(\Gamma_m^J)$ by the isomorphism σ in Theorem 3.2.

In this article we call $E_{k-1/2}^{(m)}$ *the generalized Cohen-Eisenstein series.* Indeed, when $m = 1$, $E_{k-1/2}^{(1)}$ coincides with the Cohen-Eisenstein series introduced in [Co 75].

The following proposition is shown when $r = 1$ in an adelic setting in Kawamura [Ka 08]. For more general r, we have the following.

Proposition 4.1. *Let $k > m+2$ be an even integer, r a positive integer and p a prime. We assume p^2 does not divide r, then the Jacobi-Eisenstein series $E_{k,r}^{(m)}$ of index r is an eigenform with respect to Hecke operators $\Gamma_m^J k_{s,p} \Gamma_m^J$ $(0 \le s \le m)$.*

Proof. The assertion can be proved in the same manner as in Mizumoto [Mi 97, Prop. 6.3]. However, in our case, we need a modification for the Jacobi groups.

We have

$$\begin{aligned} E_{k,r}^{(m)}|\Gamma_m^J k_{s,p}\Gamma_m^J &= p^{(m+1)(k-m)} \sum_{\gamma_1\in\Gamma_m^J\backslash\Gamma_m^J k_{s,p}\Gamma_m^J} \sum_{\gamma_2\in\Gamma_{0,m}^J\backslash\Gamma_m^J} 1|_{k,r}\gamma_2\gamma_1 \\ &= p^{(m+1)(k-m)} \sum_{\gamma\in\Gamma_{0,m}^J\backslash\Gamma_m^J k_{s,p}\Gamma_m^J} 1|_{k,r}\gamma. \end{aligned}$$

To describe a complete set of representatives of $\Gamma_{0,m}^J\backslash\Gamma_m^J k_{s,p}\Gamma_m^J$ we introduce the following two sets:

$$\mathcal{M}_0 := \left\{ \begin{pmatrix} A & B \\ 0 & p^{2t}A^{-1} \end{pmatrix} \,\middle|\, \left(\begin{pmatrix} A & B \\ 0 & p^{2t}A^{-1} \end{pmatrix}, p \right) \in \Gamma_m^J k_{s,p}\Gamma_m^J \right\},$$

and

$$\mathcal{T}_0 := \left\{ ([\lambda,0,0]) \left(\begin{pmatrix} A & B \\ 0 & p^{2t}A^{-1} \end{pmatrix}, p \right) ([0,\mu,0]) \,\middle|\, \begin{pmatrix} A & B \\ 0 & p^{2t}A^{-1} \end{pmatrix},\ \lambda,\ \mu \right\},$$

where in the definition of the set $\mathcal{T}_0$, matrix $\begin{pmatrix} A & B \\ 0 & p^{2t}A^{-1} \end{pmatrix}$ runs over a complete set of representatives of $\Gamma_{0,m}\backslash\mathcal{M}_0$, and λ (resp. μ) runs over a complete set of representatives of $\mathbb{Z}^m/(p^tA^{-1}\mathbb{Z}^m\cap\mathbb{Z}^m)$ (resp. $\mathbb{Z}^m/(pA^{-1}\mathbb{Z}^m\cap\mathbb{Z}^m)$). Then

$$\mathcal{T} \ := \ \{MM' \mid M\in\mathcal{T}_0,\ M'\}$$

is a complete set of representatives of $\Gamma_{0,m}^J\backslash\Gamma_m^J k_{s,p}\Gamma_m^J$, where M' runs over a complete set of representatives of $\Gamma_{0,m}^J\backslash\Gamma_m^J$. This fact follows from a straightforward calculation.

Now we shall describe a complete set of representatives of $\Gamma_{0,m}\backslash\mathcal{M}_0$ which appears in the definition of $\mathcal{T}_0$. We put

$$\mathfrak{A} := \left\{ A\in \mathrm{GL}_m(\mathbb{Q})\cap \mathrm{M}_m(\mathbb{Z}) \,\middle|\, \begin{pmatrix} A & * \\ 0 & p^{2t}A^{-1} \end{pmatrix} \in\mathcal{M}_0 \right\}.$$

For $A\in\mathfrak{A}$, we set $\mathfrak{B}_A := \mathrm{Sym}_m(\mathbb{Z})^tA^{-1}\cap M_m(\mathbb{Z})$, where $\mathrm{Sym}_m(\mathbb{Z})$ denotes the set of all symmetric matrices of size $m\times m$ of entries in $\mathbb{Z}$. We put

$$\mathcal{M}_0' := \left\{ \begin{pmatrix} A & B \\ 0 & p^{2t}A^{-1} \end{pmatrix} \in \mathrm{GSp}_m^+(\mathbb{Z}) \,\middle|\, A,\ B \right\},$$

where A runs over a complete set of representatives of $\mathrm{GL}_m(\mathbb{Z})\backslash\mathfrak{A}$, and B runs over a complete set of representatives of $\mathfrak{B}_A/p^2\mathfrak{B}_A$.

Then $\mathcal{M}_0'$ is a complete set of representatives of $\Gamma_{0,m}\backslash\mathcal{M}_0$. We remark that $\mathcal{M}_0'$ is a finite set, hence $\mathcal{T}_0$ is also a finite set.

Thus we have

$$E_{k,r}^{(m)}|\Gamma_m^J k_{s,p}\Gamma_m^J = p^{(m+1)(k-m)} \sum_{M'\in\Gamma_{0,m}^J\backslash\Gamma_m^J} \left(\sum_{M\in\mathcal{T}_0} 1|_{k,r}M\right)|_{k,r}M'.$$

Here we get

$$\sum_{M\in\mathcal{T}_0} 1|_{k,r}M = \sum_{A,\lambda,\mu'}\sum_{B\in\mathfrak{B}_A/p^2\mathfrak{B}_A} 1|_{k,r}([\lambda,0,0])\left(\begin{pmatrix} A & B \\ 0 & p^{2t}A^{-1}\end{pmatrix},p\right)([0,\mu,0]),$$

where in the above first summation, A, λ and μ run over all representatives of $\mathrm{GL}_m(\mathbb{Z})\backslash\mathfrak{A}$, of $\mathbb{Z}^m/(p^tA^{-1}\mathbb{Z}^m\cap\mathbb{Z}^m)$ and of $\mathbb{Z}^m/(pA^{-1}\mathbb{Z}^m\cap\mathbb{Z}^m)$, respectively. Now, by the definition of the slash operator, we have

$$\begin{aligned}
&\sum_{B\in\mathfrak{B}_A/p^2\mathfrak{B}_A} 1|_{k,r}([\lambda,0,0])\left(\begin{pmatrix} A & B \\ 0 & p^{2t}A^{-1}\end{pmatrix},p\right)([0,\mu,0]) \\
&= c'\,e\left(\frac{r}{p^2}{}^t\lambda A\tau {}^tA\lambda + \frac{2r}{p}{}^t\lambda Az + \frac{2r}{p}{}^t\lambda A\mu\right)\sum_{B\in\mathfrak{B}_A/p^2\mathfrak{B}_A} e\left(\frac{r}{p^2}{}^t\lambda B{}^tA\lambda\right),
\end{aligned}$$

where c' is a certain constant which depends only on the choice of A and of k. Since r is not divisible by p^2, we obtain $\displaystyle\sum_{B\in\mathfrak{B}_A/p^2\mathfrak{B}_A} e\left(\frac{r}{p^2}{}^t\lambda B{}^tA\lambda\right) = 0$ if $A^t\lambda\notin p\mathbb{Z}^m$. Thus we get

$$\sum_{M\in\mathcal{T}_0} 1|_{k,r}M = \sum_A c''\,e\left(\frac{r}{p^2}{}^t\lambda A\tau{}^tA\lambda + \frac{2r}{p}{}^t\lambda Az\right) = \sum_A c''\,1|_{k,r}([{}^tA\lambda/p,0,0]),$$

where in the summations c'' is a suitable constant, and matrices A runs over a complete set of representatives of $\mathrm{GL}_m(\mathbb{Z})\backslash\mathfrak{A}$, and the element $\lambda\in p^tA^{-1}\mathbb{Z}^m\cap\mathbb{Z}^m$ is uniquely determined by the choice of A. Thus

$$\begin{aligned}
E_{k,r}^{(m)}|\Gamma_m^J k_{s,p}\Gamma_m^J &= p^{(m+1)(k-m)}\sum_{M'\in\Gamma_{0,m}^J\backslash\Gamma_m^J}\sum_A c''\,1|_{k,r}([{}^tA\lambda/p,0,0])M' \\
&= p^{(m+1)(k-m)}\sum_A c''\sum_{M''\in\Gamma_{0,m}^J\backslash\Gamma_m^J} 1|_{k,r}M''.
\end{aligned}$$

Since $\displaystyle\sum_{M''\in\Gamma_{0,m}^J\backslash\Gamma_m^J} 1|_{k,r}M'' = E_{k,r}^{(m)}$, we conclude the proposition. □

Corollary 4.2. *The generalized Cohen-Eisenstein series $E_{k-1/2}^{(m)}$ is a Hecke eigenform for the action of any Hecke operators $\hat{T}_s(p^2)$ defined in § 2.2.*

Proof. It follows from Theorem 3.2 and Proposition 4.1. □

The Eisenstein series $E_k^{(m+1)}(Z)$ has the Fourier-Jacobi expansion:

$$E_k^{(m+1)}(Z) \;=\; \sum_{r\geq 0} e_{k,r}^{(m)}(\tau_1,z)e(r\tau') \tag{4.2}$$

where $Z = \begin{pmatrix} \tau_1 & z \\ {}^tz & \tau' \end{pmatrix} \in \mathfrak{H}_{m+1}$, $\tau_1 \in \mathfrak{H}_m$, $\tau' \in \mathfrak{H}_1$, and $e_{k,r}^{(m)}(\tau_1,z) \in J_{k,r}(\Gamma_m^J)$.

We use the following lemma to show Theorem 0.1. This lemma is a special case of a theorem obtained by S. Boecherer [Bo 83].

Lemma 4.3 ([Bo 83]). *For any even integer $k > m+2$ we have*

$$e_{k,1}^{(m)}(\tau,z) = E_{k,1}^{(m)}(\tau,z).$$

Proof. See [Bo 83, Satz 7] (see also [Ya 86, Theorem 5.5]). □

5. Yamazaki's formula

In this section we introduce a formula shown by Yamazaki [Ya 86]. This is a formula of a relation among Fourier-Jacobi coefficients of Siegel-Eisenstein series. This section owe to [Ya 86], [Ya 89].

We define the map $\rho : \mathrm{GSp}_{2n-1}^+(\mathbb{Z}) \to \mathrm{GSp}_{2n}^+(\mathbb{Z})$ by

$$\rho(M) := \begin{pmatrix} A & 0 & B & 0 \\ 0 & \nu(M) & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{for } M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{GSp}_{2n}^+(\mathbb{Z}),$$

where $\nu(M)$ is the similitude of M.

For $M \in \mathrm{GSp}_{2n-1}^+(\mathbb{Z})$ and for $\phi \in J_{k,r}(\Gamma_{2n-1,1}^J)$ we recall the action of double coset $\Gamma_{2n-1}^J\rho(M)\Gamma_{2n-1}^J$ (cf. § 3.2):

$$\phi|\Gamma_{2n-1}^J\rho(M)\Gamma_{2n-1}^J = \nu(M)^{nk-(2n-1)n} \sum_{M_i\in\Gamma_{2n-1}\backslash\Gamma_{2n-1}M\Gamma_{2n-1}} \phi|_{k,r}\rho(M_i),$$

where we use the disjoint decomposition

$$\Gamma_m^J\rho(M)\Gamma_m^J \;=\; \bigcup_{M_i\in\Gamma_{2n-1}\backslash\Gamma_{2n-1}M\Gamma_{2n-1}} \Gamma_m^J\,\rho(M_i).$$

The form $\phi|\Gamma_{2n-1}^J\rho(M)\Gamma_{2n-1}^J$ belongs to $J_{k,\nu(M)r}(\Gamma_{2n-1,1}^J)$.

We put two Hecke operators $T^J(p)$ and $T_{0,2n}^J(p^2)$ which are defined by the action of double cosets $\Gamma_{2n-1}^J\rho\left(\left(\begin{smallmatrix} 1_{2n-1} & 0 \\ 0 & p\cdot 1_{2n-1} \end{smallmatrix}\right)\right)\Gamma_{2n-1}^J$ and $\Gamma_{2n-1}^J\rho((p\cdot 1_{4n-2}))\Gamma_{2n-1}^J$, respectively.

For Siegel-Eisenstein series, the following relation is shown by Yamazaki.

Theorem 5.1 ([Ya 86]). *Let $k' > m+2$, and let $e_{k',r}^{(m)}$ be the r-th Fourier-Jacobi coefficient of Siegel-Eisenstein series $E_{k'}^{(m+1)}$ (see (4.2)). Then we have*

$$e_{k',r}^{(m)} \;=\; e_{k',1}^{(m)}|_{k',1}D_m(r),$$

where the Hecke operator $D_m(r)$ *is defined through the formal Dirichlet series:*

$$\sum_{r>0}\frac{D_m(r)}{r^s} :=$$
$$\prod_{p:prime}\left(1-\left(\prod_{1<i\le m}(1+p^{k'-i})^{-1}\right)T^J(p)p^{\frac{1}{2}(m-1)k'-s}+T^J_{0,m}(p^2)p^{m(m+1)-1-2s}\right)^{-1}.$$

It is remarked in [Ya 86] that if $m=1$ then the above relation coincides so-called Maass relation.

For odd degree $m=2n-1$, we have the following corollary by using the operator $D_{2n-1}(r,\{\alpha_p\}_p)$ defined in Theorem 0.2.

Corollary 5.2. *Let* $k'>n+1$ *be an positive integer which satisfies* $k'+n\in 2\mathbb{Z}$*. Then we have*

$$e^{(2n-1)}_{k'+n,r} \;=\; e^{(2n-1)}_{k'+n,1}|_{k'+n,1}D_{2n-1}(r,\{p^{k'-\frac{1}{2}}\}_p)\ .$$

Proof. The Satake parameter of Eisenstein series $E^{(1)}_{2k'}$ is $\{\alpha_p^{\pm}\}_p=\{p^{\pm(k'-1/2)}\}_p$. This corollary follows directly from the identity

$$\prod_{1<i\le 2n-1}(1+p^{k'+n-i})^{-1}$$
$$=p^{(1-n)k'-\frac{1}{2}(n-1)(n-2)}\prod_{1\le i\le n-1}(1+p^{(k'-1/2)+1/2-i})^{-1}(1+p^{-(k'-1/2)+1/2-i})^{-1}.$$

□

6. Proof of Theorems

In this section we shall show Theorem 0.1 and 0.2 stated in § 0.2. We use the same notation in § 0.2.

The following Lemma shown in [Ik 01, Lemma 10.1] plays an important rule in this section.

Lemma 6.1 ([Ik 01])**.** *Let* $F(\{X_p\}_p)\in\mathbb{C}[X_2+X_2^{-1},X_3+X_3^{-1},X_5+X_5^{-1},...]$*. If* F *satisfies* $F(\{p^{k-1/2}\}_p)=0$ *for sufficiently many* $k\in\mathbb{Z}$*, then* $F\equiv 0$

Proof. We write $F(\{X_p\}_p)=\sum_n a_nX_n$, where $X_n=\prod_p X_p^{\mathrm{ord}_p(n)}$. Because $F(\{p^{k-1/2}\}_p)=\sum_n a_n n^{k-1/2}=0$ for sufficiently many $k\in\mathbb{Z}$, we obtain $a_n=0$ for all n. □

We shall show Theorem 0.1

Proof of Theorem 0.1. We proceed as in [Ik 01, p.664-665]. First, we shall show that the form $\Psi^{(2n-1)}(g)$ is a Hecke eigenform.

Let $\mathbb{L}_{2n-1,0}$ be the set defined in §3.3. Let $M \in \mathbb{L}_{2n-1,0}$ be a positive-definite symmetric matrix. There exist $N \in \mathfrak{N}_{2n-1}^{+}$ and $R \in \mathbb{Z}^{2n-1}$ which satisfy $M = 4\,N - R^tR$. We put $M_1 = \begin{pmatrix} N & \frac{1}{2}R \\ \frac{1}{2}{}^tR & 1 \end{pmatrix} \in \mathfrak{N}_{2n,1}^{+}$. Then, by virtue of Theorem 3.2, M-th Fourier coefficient of $\Psi^{(2n-1)}(g)$ is

$$c(\delta_{M_1}) f_{M_1}^{k-\frac{1}{2}} \prod_{q|f_{M_1}} \tilde{F}_q\left[M_1, \alpha_q\right]. \tag{6.1}$$

Let $\hat{T}_s(p^2)$ be a Hecke operator introduced in §2.2 and $\{M'_v\}_v$ a complete set of representatives of $\Gamma_0^{(2n-1)}(4)\backslash\Gamma_0^{(2n-1)}(4)k'_{s,p}\Gamma_0^{(2n-1)}(4)$. In particular we can take M'_v as an upper triangle matrix $M'_v = \begin{pmatrix} * & * \\ 0_{2n-1} & D_v \end{pmatrix}$, where $D_v \in M_{2n-1}(\mathbb{Z}) \cap GL_{2n-1}(\mathbb{Q})$. Then, by straightforward calculation, it is shown that the M-th Fourier coefficient of $\Psi^{(2n-1)}(g)|\hat{T}_s(p^2)$ is

$$c(\delta_{M_1}) f_{M_1}^{k-\frac{1}{2}} \sum_{v} \beta_v \prod_{q|f_{M_{1,v}}} \tilde{F}_q\left[M_{1,v}, \alpha_q\right], \tag{6.2}$$

where, in the above sum, β_v are certain numbers that are independent of the choice of $\Psi^{(2n-1)}(g)$ and of k, and we set the matrix $M_{1,v} := p^{-2}M_1\left[\begin{pmatrix} D_v & \\ & p \end{pmatrix}\right]$, and where D_v is the right lower part of M'_v. In the above product q runs over all primes which satisfy $q|f_{M_{1,v}}$. In the case of $M_{1,v} \notin \mathfrak{N}_{2n}^{+}$, we regard $\prod \tilde{F}_q\left[M_{1,v}, \alpha_q\right]$ as 0. To deduce the above identity (6.2), we used the identities $f_{M_{1,v}} = p^{-2n+1}\det(D_v) f_{M_1}$ and $\delta_{M_{1,v}} = \delta_{M_1}$.

Let $k' > n+1$ be an integer such that $k'+n \equiv 0 \pmod 2$. By using the lemma 4.3 and (6.1), we know that the M-th Fourier coefficient of $E_{k'+n-1/2}^{(2n-1)}$ is

$$h_{k'+1/2}(\delta_{M_1}) f_{M_1}^{k'-\frac{1}{2}} \prod_{q|f_{M_1}} \tilde{F}_q\left[M_1, q^{k'-1/2}\right],$$

where $h_{k'+1/2}(\delta_{M_1})$ is the δ_{M_1}-th Fourier coefficient of Cohen-Eisenstein series $E_{k'+1/2}^{(1)}$ of weight $k'+1/2$. Moreover, the M-th Fourier coefficient of $E_{k'+n-1/2}^{(2n-1)}|\hat{T}_s(p^2)$ is

$$h_{k'+1/2}(\delta_{M_1}) f_{M_1}^{k'-\frac{1}{2}} \sum_{M'_v} \beta_v \prod_{q|f_{M_{1,v}}} \tilde{F}_q\left[M_{1,v}, q^{k'-1/2}\right], \tag{6.3}$$

where M'_v, β_v and $M_{1,v}$ are the same symbols in (6.2). Since $E_{k'+n-1/2}^{(2n-1)}(\tau)$ is a Hecke eigenform (see Corollary 4.2) and since $h_{k'+1/2}(\delta_{M_1}) \neq 0$, we have

$$\sum_{M'_v} \beta_v \prod_{q|f_{M_{1,v}}} \tilde{F}_q\left[M_{1,v}, q^{k'-1/2}\right] \;=\; \eta_{k',p,s} \prod_{q|f_{M_1}} \tilde{F}_q\left[M_1, q^{k'-1/2}\right],$$

where $\eta_{k',p,s}$ is a certain constant which depends only on the choice of three numbers k', p and s. In particular, if we replace M_1 with $B_0 \in \mathfrak{N}^+_{2n-1,1}$ which satisfies $\tilde{F}_p(B_0, X_p) = 1$, then we know that there exists a Laurent-Polynomial $\Phi(\tilde{T}_s(p^2), X_p)$ which satisfies $\eta_{k',p,s} = \Phi(\tilde{T}_s(p^2), p^{k'-1/2})$ for sufficiently many k'. Hence, by using Lemma 6.1, we have

$$\sum_{M'_v} \beta_v \prod_{q|f_{M_{1,v}}} \tilde{F}_q\left[M_{1,v}, X_q\right] = \Phi(\tilde{T}_s(p^2), X_p) \prod_{q|f_{M_1}} \tilde{F}_q\left[M_1, X_q\right]. \tag{6.4}$$

Thus we have the following identity

$$c(\delta_{M_1}) f_{M_1}^{k-\frac{1}{2}} \sum_{M'_v} \beta_v \prod_{q|f_{M_{1,v}}} \tilde{F}_q\left[M_{1,v}, \alpha_q\right] = \Phi(\tilde{T}_s(p^2), \alpha_p)\, c(\delta_{M_1}) f_{M_1}^{k-\frac{1}{2}} \prod_{q|f_{M_1}} \tilde{F}_q\left[M_1, \alpha_q\right] \ .$$

Hence the form $\Psi^{(2n-1)}(g)$ is a Hecke eigenform.

Next we shall show the expression of $L(s, \Psi^{(2n-1)}(g))$ stated in Theorem 0.1. By using Zharkovskaya's theorem for Siegel modular forms of half-integral weight (cf. Theorem 2.1) we obtain

$$\begin{aligned} L(s, E^{(2n-1)}_{k'+n-\frac{1}{2}}) &= \prod_{i=0}^{2n-2} L(s + k' - n + 1/2 + i, E^{(1)}_{2(k'-n+1+i)}) \\ &= \prod_{i=0}^{2n-2} \zeta(s + k' - n + 1/2 + i)\zeta(s - k' + n - 1/2 - i) \\ &= \prod_{i=0}^{2n-2} \zeta(s + k' + n - 3/2 - i)\zeta(s - k' + n - 1/2 - i) \\ &= \prod_{i=1}^{2n-1} L(s + k' + n - 1/2 - i, E^{(1)}_{2k'}), \end{aligned}$$

where $L(s, E^{(1)}_{2k'}) := \prod_{\substack{p \\ p\neq 2}} \left\{(1-p^{-s})(1-p^{2k'-1-s})\right\}^{-1}$ is the Hecke L-function of Eisenstein series of weight $2k'$ excluding the Euler 2-factor. In particular, the Euler p-factor of $L(s, E^{(2n-1)}_{k'+n-\frac{1}{2}})$ and the Euler p-factor of $\prod_{i=1}^{2n-1} L(s + k' + n - 1/2 - i, E^{(1)}_{2k'})$ coincides for any odd prime p.

Now, we define a polynomial $\gamma_j = \gamma_j(x_1, ..., x_{2n-1})$ by $\prod_{i=1}^{2n-1}(1 - x_i t)(x_i^{-1} t) = \sum_{j=0}^{4n-2} \gamma_j t^j$, and we fix an odd prime p. Then there exists a Hecke operator $T_j \in \hat{L}^{2n-1}_p$ which corresponds to γ_j by the Satake isomorphism (cf. [Zh 84, Proposition 4.1].) For the Hecke operator T_j we define the Laurent-Polynomial $\Phi(T_j, X_p)$ in the same manner as in the identity (6.4). Namely the eigenvalue of $E^{(2n-1)}_{k'+n-1/2}$ for T_j is $\Phi(T_j, p^{k'-1/2})$. Hence, by using Lemma 6.1, the above expression of $L(s, E^{(2n-1)}_{k'+n-\frac{1}{2}})$ gives an explicit structure of $\Phi(T_j, X_p)$. On the other hand, the eigenvalue of $\Psi^{(2n-1)}(g)$ for T_j is $\Phi(T_j, \alpha_p)$. Therefore we have the identity $L(s, \Psi^{(2n-1)}(g)) = \prod_{i=1}^{2n-1} L(s+k+n-1/2-i, g)$. Thus we conclude Theorem 0.1. □

Next we shall show Theorem 0.2.

Proof of Theorem 0.2. Let $e^{(2n-1)}_{k'+n,r}$ be the r-th Fourier-Jacobi coefficient of the Eisenstein series $E^{(2n)}_{k'+n}$ defined in (4.2). Let $M_r = \left(\begin{smallmatrix} N & R \\ {}^tR & r \end{smallmatrix}\right) \in \mathfrak{N}^+_{2n-1,r}$. We denote by $A_{k'+n}(M_r)$ the (N,R)-th Fourier coefficient of $e^{(2n-1)}_{k'+n,r}$. Then we have

$$A_{k'+n}(M_r) \;=\; h_{k'+1/2}(\delta_{M_r}) f_{M_r}^{k'-1/2} \prod_{p|f_{M_r}} \tilde{F}_p\left[M_r, p^{k'-1/2}\right].$$

On the other hand, by virtue of Corollary 5.2, $A_{k'+n}(M_r)$ is equal to the (N,R)-th Fourier coefficient of $(e^{(2n-1)}_{k'+n,1}|_{k'+n,1} D_{2n-1}(r, \{p^{k'-\frac{1}{2}}\}))$. Thus there exist constants $\{\gamma_v\}_v$ and matrices $\{M_{r,v}\}_v$ which satisfy

$$A_{k'+n}(M_r) \;=\; h_{k'+1/2}(\delta_{M_r}) f_{M_r}^{k'-1/2} \sum_v \gamma_v \prod_{p|f_{M_{r,v}}} \tilde{F}_p\left[M_{r,v}, p^{k'-1/2}\right],$$

By straightforward calculation, we know that constants $\{\gamma_v\}_v$ and matrices $\{M_{r,v}\}_v$ are independent of the choice of k'.

Hence, by using Lemma 6.1, we have

$$\sum_v \gamma_v \prod_{p|f_{M_{r,v}}} \tilde{F}_p\left[M_{r,v}, X_p\right] \;=\; \prod_{p|f_{M_r}} \tilde{F}_p\left[M_r, X_p\right].$$

Therefore, putting $X_p = \alpha_p$ in the above identity and multiplying both sides by $c(\delta_{M_r}) f_{M_r}^{k-1/2}$, we conclude $\phi_r = \phi_1|_{k+n,1} D_{2n-1}(r, \{\alpha_p\}_p)$. □

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