

A CONVERSE THEOREM FOR HILBERT-JACOBI FORMS

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1. Introduction and statement of results. Doi and Naganuma, see [6], constructed a lifting map from elliptic modular forms to Hilbert modular forms in the case of a real quadratic field with narrow class number one. A converse theorem for Hilbert modular forms was one of their basic tools. This gives rise to the question of constructing a lifting map in the case of Jacobi forms. Here we do the first step in that direction and prove a converse theorem for Hilbert-Jacobi forms.

Studying the connection between functions that satisfy certain transformation laws and the functional equation of their associated L-functions has value on its own and a long history. In a celebrated paper, see [9], Hecke showed that the automorphy of a cusp form with respect to $\mathrm{SL}_2(\mathbf{Z})$ is equivalent to the functional equation of its associated L-functions. That only one functional equation is needed is in a way atypical and highly depends on the fact that $\mathrm{SL}_2(\mathbf{Z})$ is generated by the matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. This situation already changes if one considers cusp forms with respect to a subgroup of $\mathrm{SL}_2(\mathbf{Z})$ which have a character. In this case the functional equation of twists is required, see [18].

Hecke's work has inspired an astonishing number of people and a lot of generalizations of his "converse theorem" have been made, e.g., generalizations to Hilbert modular forms as mentioned above, see [6], Siegel modular forms, see [2, 10], or Jacobi forms, see [14, 15]. Maass showed an analogue of Hecke's result for nonholomorphic modular forms, see [13]. He proved that these correspond to certain L-functions in quadratic fields. An outstanding generalization of a converse theorem for $\mathrm{GL}(n)$ was done by Jacquet and Langlands for $n = 2$, see [11],

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Jacquet, Piatetski-Shapiro and Shalika for $n = 3$, see [12], and Cogdell and Piatetski-Shapiro for general n , see [5].

In this paper, we prove a converse theorem for Hilbert-Jacobi cusp forms over a totally real number field K of degree $g := [K : \mathbf{Q}]$ with discriminant D_K and narrow class number 1. The case $g = 1$, i.e., Jacobi forms over \mathbf{Q} as considered by Eichler and Zagier, see [7], is treated in two interesting papers by Martin, see [14, 15]. To describe our result, we consider functions $\phi(\tau, z)$ from $\mathbf{H}^g \times \mathbf{C}^g$ into \mathbf{C} that have a Fourier expansion with certain conditions on the Fourier coefficients, see (3.4), (3.5) and (3.6). We show that ϕ is a Hilbert-Jacobi cusp form (for the definition see Section 2) if and only if certain Dirichlet series $\mathcal{L}(s, \phi, r, \chi_{m,\nu})$, see (3.9), satisfy functional equations. More precisely, we show the following.

Theorem 1.1. *Let k be an integer, and let $m \in \mathfrak{d}_K^{-1}$ be the inverse different from K . A function ϕ satisfying (3.3), (3.4), (3.5) and (3.6) is a Hilbert-Jacobi cusp form of weight k and index m if and only if for all ν satisfying (3.1) and (3.2) and for all $r \in \mathfrak{d}_K^{-1}/(2m\mathcal{O}_K)$ the functions $\mathcal{L}(s, \phi, r, \chi_{m,\nu})$, see Definition 3.2, have analytic continuations to the whole complex plane, are bounded in every vertical strip and satisfy the functional equations*

$$\begin{aligned} \mathcal{L}(s, \phi, r, \chi_{m,\nu}) &= \frac{1}{\sqrt{D_K}} i^{-kg} \mathbf{N}(2m)^{-1/2} \\ &\times \sum_{\mu \in (\mathfrak{d}_K^{-1}/2m\mathcal{O}_K)} e_{2m}(-\mu r) \mathcal{L}(k - s - 1/2, \phi, \mu, \chi_{m,-\nu}), \end{aligned}$$

see Section 2 for the definition of \mathbf{N} and $e_{2m}(\cdot)$.

We proceed as follows. In Section 2 we recall basic facts about Hilbert-Jacobi cusp forms. In particular, we show that these have a theta decomposition, see (2.3), where the involved theta series satisfy some transformation law, see Lemma 2.1. Section 3 deals with certain characters of Hecke type and the Dirichlet series needed for the converse theorem. In Section 4, we prove Theorem 1.1.

2. Basic facts about Hilbert-Jacobi cusp forms. We let K be a totally real number field of degree $g := [K : \mathbf{Q}]$ and denote by \mathcal{O}_K ,

\mathcal{O}_K^\times , \mathfrak{d}_K , and D_K its ring of integers, units, different, and discriminant, respectively. We denote the j th embedding, $1 \leq j \leq g$, of an element $l \in K$ by $l^{(j)}$. An element $l \in K$ is said to be totally positive, $l > 0$, if all its embeddings into \mathbf{R} are positive.

Let us now briefly recall some basic facts about Hilbert-Jacobi cusp forms, see also [16]. We put $\Gamma_K := \mathrm{SL}_2(\mathcal{O}_K)$. Let the Hilbert-Jacobi group be defined as the set $\Gamma_K^J := \Gamma_K \times (\mathcal{O}_K \times \mathcal{O}_K)$, with the group multiplication

$$\gamma_1 \cdot \gamma_2 := \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}, (\lambda_1, \mu_1) \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} + (\lambda_2, \mu_2) \right),$$

where we put

$$\gamma_i := \left(\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}, (\lambda_i, \mu_i) \right) \in \Gamma_K^J, \quad \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \Gamma_K,$$

and $(\lambda_i, \mu_i) \in \mathcal{O}_K \times \mathcal{O}_K$. The Hilbert-Jacobi group is generated by the following three types of elements

$$(2.1) \quad \begin{aligned} & \left(\begin{pmatrix} \varepsilon & \lambda \\ 0 & \varepsilon^{-1} \end{pmatrix}, (0, 0) \right), \\ & \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, (0, 0) \right), \end{aligned}$$

and

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, (\lambda, \mu) \right),$$

where $\lambda, \mu \in \mathcal{O}_K$ and $\varepsilon \in \mathcal{O}_K^\times$, see [1, 4, 17].

The Hilbert-Jacobi group acts on $\mathbf{H}^g \times \mathbf{C}^g$ (\mathbf{H} is the usual upper half-plane) by

$$\begin{aligned} & \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu) \right) \circ (\tau, z) \\ & := \left(\left(\frac{a^{(1)}\tau_1 + b^{(1)}}{c^{(1)}\tau_1 + d^{(1)}}, \dots, \frac{a^{(g)}\tau_g + b^{(g)}}{c^{(g)}\tau_g + d^{(g)}} \right), \right. \\ & \quad \left. \left(\frac{z_1 + \lambda^{(1)}\tau_1 + \mu^{(1)}}{c^{(1)}\tau_1 + d^{(1)}}, \dots, \frac{z_g + \lambda^{(g)}\tau_g + \mu^{(g)}}{c^{(g)}\tau_g + d^{(g)}} \right) \right), \end{aligned}$$

where

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu) \right) \in \Gamma_K^J,$$

$\tau = (\tau_1, \dots, \tau_g) \in \mathbf{H}^g$ and $z = (z_1, \dots, z_g) \in \mathbf{C}^g$. Throughout this paper, we write $\tau = u + iv$, $z = x + iy$, $\tau_j = u_j + iv_j$ and $z_j = x_j + iy_j$, $1 \leq j \leq g$.

Let $k \in \mathbf{N}$, $m \in \mathfrak{d}_K^{-1}$ totally positive,

$$\gamma = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu) \right) \in \Gamma_K^J,$$

and a function $\phi : \mathbf{H}^g \times \mathbf{C}^g \rightarrow \mathbf{C}$. Then we define

$$\begin{aligned} \phi|_{k,m}\gamma(\tau, z) &:= \mathbf{N}(c\tau + d)^{-k} \\ &\cdot e\left(-\left(\frac{cm(z + \lambda\tau + \mu)^2}{c\tau + d} + m\tau\lambda^2 + 2m\lambda z\right)\right) \\ &\cdot \phi(\gamma \circ (\tau, z)), \end{aligned}$$

where for $\alpha \in K$ and for $z \in \mathbf{C}^g$, we define $\mathbf{N}(\alpha z) := \prod_{j=1}^g (\alpha^{(j)} z_j)$, $\text{tr}(az) := \sum_{j=1}^g a^{(j)} z_j$ and $e(\alpha z) := e^{2\pi i \text{tr}(\alpha z)}$.

A holomorphic function $\phi : \mathbf{H}^g \times \mathbf{C}^g \rightarrow \mathbf{C}$ is called a *Hilbert-Jacobi cusp form* of weight k and index m if $\phi|_{k,m}\gamma(\tau, z) = \phi(\tau, z)$ for all $\gamma \in \Gamma_K^J$ and if it has a Fourier expansion of the form

$$\sum_{\substack{n, r \in \mathfrak{d}_K^{-1} \\ 4nm - r^2 > 0}} c(n, r) e(n\tau + rz).$$

In [16], m is chosen to be in \mathcal{O}_K , but our choice $m \in \mathfrak{d}_K^{-1}$ seems more natural since in this way the coefficients of Hilbert-Siegel modular forms are examples for Jacobi forms as in the classical case.

If ϕ is a Hilbert-Jacobi cusp form, then the transformation $(\tau, z) \rightarrow (\tau, z + \lambda\tau + \mu)$ leads to

$$(2.2) \quad c(n, r) = c(n + \lambda r + \lambda^2 m, r + 2\lambda m), \quad \text{for all } \lambda \in \mathcal{O}_K.$$

From this, we can deduce that

$$(2.3) \quad \phi(\tau, z) = \sum_{r \in (\mathfrak{d}_K^{-1}/2m\mathcal{O}_K)} f_r(\tau) \vartheta_{m,r}(\tau, z),$$

where, for $r \in (\mathfrak{d}_K^{-1}/2m\mathcal{O}_K)$, we define

$$(2.4) \quad f_r(\tau) := \sum_{\substack{n \in \mathfrak{d}_K^{-1} \\ 4nm - r^2 > 0}} c(n, r) e_{4m}((4nm - r^2)\tau),$$

$$(2.5) \quad \vartheta_{m,r}(\tau, z) := \sum_{\lambda \in \mathcal{O}_K} e_{4m} \left((r + 2\lambda m)^2 \tau + 4m(r + 2\lambda m)z \right),$$

and where, for $\alpha, \beta \in K$, $\beta \neq 0$ and $z \in \mathcal{C}^g$, we define $e_\beta(\alpha z) := e(\beta^{-1}\alpha z)$.

The theta series $\vartheta_{m,r}$ satisfy the following transformation law.

Lemma 2.1. *If $m \in \mathfrak{d}_K^{-1}$ is totally positive and $\mu \in (\mathfrak{d}_K^{-1}/2m\mathcal{O}_K)$, then we have*

$$\begin{aligned} \vartheta_{m,\mu} \left(-\frac{1}{\tau}, \frac{z}{\tau} \right) &= \frac{1}{\sqrt{D_K}} \mathbf{N} \left((\tau/i)^{1/2} \right) \cdot \mathbf{N} (2m)^{-1/2} \\ &\quad \cdot e \left(\frac{m \cdot z^2}{\tau} \right) \sum_{r \in (\mathfrak{d}_K^{-1}/2m\mathcal{O}_K)} e_{2m}(-\mu r) \vartheta_{m,r}(\tau, z), \end{aligned}$$

where we put $(\tau/i)^{1/2} := ((\tau_1/i)^{1/2}, \dots, (\tau_g/i)^{1/2})$, and we take the principal value of the square root, namely, $-\pi/2 < \arg(w) \leq \pi/2$ for $w \in \mathbf{C}$.

From Lemma 2.1, we obtain

Corollary 2.2. *A function $\phi : \mathbf{H}^g \times \mathbf{C}^g$ having a decomposition of the form (2.3) satisfies*

$$\phi \left(-\frac{1}{\tau}, \frac{z}{\tau} \right) = \mathbf{N}(\tau)^k e \left(\frac{mz^2}{\tau} \right) \phi(\tau, z)$$

if and only if

$$(2.6) \quad f_r(\tau) = \frac{1}{\sqrt{D_K}} i^{-kg} \mathbf{N} \left((\tau/i)^{1/2-k} \right) \mathbf{N} (2m)^{-1/2} \\ \times \sum_{\mu \in (\mathfrak{d}_K^{-1}/2m\mathcal{O}_K)} e_{2m}(-\mu r) f_\mu \left(-\frac{1}{\tau} \right),$$

for all $r \in (\mathfrak{d}_K^{-1}/2m\mathcal{O}_K)$. In particular, if ϕ is a Hilbert-Jacobi cusp form, then ϕ satisfies (2.6).

Exactly as in the case of elliptic modular forms, one can show:

Lemma 2.3. *Assume that ϕ is a Hilbert-Jacobi cusp form, with f_r defined as in (2.4). Let c_1 be a positive real number, and let S be the subset of \mathbf{H}^g such that for all $\tau \in S$ the components v_j , $1 \leq j \leq g$, are larger than c_1 . Then we have*

$$(2.7) \quad |f_r(\tau)| \ll_{\phi, c_1} e^{-c_2(\sum_{j=1}^g v_j)},$$

where c_2 is a positive constant, and where the constant implied in \ll_{ϕ, c_1} depends on ϕ and on c_1 .

Lemma 2.4. *If ϕ is a Hilbert-Jacobi cusp form of weight k and index m , then the function*

$$g(\tau, z) := \mathbf{N}(v)^{k/2} \exp\left(-2\pi \operatorname{tr}\left(\frac{my^2}{v}\right)\right) \phi(\tau, z)$$

is bounded on $\mathbf{H}^g \times \mathbf{C}^g$.

By using Lemma 2.4, we have the following.

Lemma 2.5. *If ϕ is a Hilbert-Jacobi cusp form of weight k and index m with Fourier coefficients $c(n, r)$, then $|c(n, r)| \ll_{\phi} \mathbf{N}(4mn - r^2)^{k/2}$.*

3. Hecke-type characters and Dirichlet series. For the remainder we assume that k is an integer. For $m \in \mathfrak{d}_K^{-1}$, we let T_m be the subgroup of \mathcal{O}_K^\times defined by

$$T_m := \{\varepsilon \in \mathcal{O}_K^\times \mid \varepsilon - 1 \in 2m\mathfrak{d}_K\}.$$

We have that $\varepsilon \in \mathcal{O}_K^\times$ is in T_m if and only if $\varepsilon r - r \in 2m\mathcal{O}_K$ for every $r \in \mathfrak{d}_K^{-1}/(2m\mathcal{O}_K)$.

We let u_1, \dots, u_{g-1} be a basis of T_m^2 , where $T_m^2 := \{\varepsilon^2 \mid \varepsilon \in T_m\}$. We take $\varepsilon_1, \dots, \varepsilon_{g-1} \in T_m$ which satisfy $\varepsilon_l^2 = u_l$ for $l = 1, \dots, g-1$. If m is not a generator of the different inverse, then T_m does not contain -1 ; hence, the ε_l are uniquely determined. If m is a generator of the different inverse, then T_m contains -1 , and we choose $\varepsilon_l > 0$ as a solution of the above equation.

For integers $N_l, 1 \leq l \leq g-1$, we choose pure imaginary solutions ν_1, \dots, ν_g which satisfy the following equations

$$(3.1) \quad \sum_{j=1}^g \nu_j = 0,$$

$$(3.2) \quad \sum_{j=1}^g \nu_j \log(u_l^{(j)}) = 2\pi i \left(N_l + \frac{1}{2} \delta_l \right),$$

where we put $\delta_l = 0$ or 1 if $\mathbf{N}(\varepsilon_l)^k = 1$ or -1 , respectively. For any integers $N_l, l = 1, \dots, g-1$, we have a solution to (3.1) and (3.2), because

$$\begin{aligned} \det \begin{pmatrix} 1 & \cdots & 1 \\ \log(u_1^{(1)}) & \cdots & \log(u_1^{(g)}) \\ \vdots & \cdots & \vdots \\ \log(u_{g-1}^{(1)}) & \cdots & \log(u_{g-1}^{(g)}) \end{pmatrix} \\ = (-1)^{g+1} g \cdot \det((\log(u_l^{(j)}))_{l,j=1,\dots,g-1}) \neq 0, \end{aligned}$$

where the last inequality can be obtained from the fact that basis elements u_l are multiplicatively independent.

For $x \in K$ and $\nu := (\nu_1, \dots, \nu_g)$ satisfying (3.1) and (3.2), we set

$$\chi_{m,\nu}(x) := \prod_{j=1}^g |x^{(j)}|^{\nu_j}.$$

To define the Dirichlet series needed, we consider functions $\phi(\tau, z)$ from $\mathbf{H}^g \times \mathbf{C}^g$ into \mathbf{C} that have a Fourier expansion of the form

$$(3.3) \quad \phi(\tau, z) = \sum_{\substack{n,r \in \mathfrak{d}_K^{-1} \\ 4nm - r^2 > 0}} c(n, r) e(n\tau + rz)$$

that is absolutely and locally uniformly convergent. We regard $c(n, r) = 0$ unless $4nm - r^2 > 0$ or unless $n, r \in \mathfrak{d}_K^{-1}$. Moreover, we demand that its Fourier coefficients satisfy

$$(3.4) \quad c(n, r) = c(n + \lambda r + \lambda^2 m, r + 2\lambda m) \quad \text{for all } \lambda \in \mathcal{O}_K,$$

$$(3.5) \quad c(\varepsilon^2 n, \varepsilon r) = \mathbf{N}(\varepsilon)^k c(n, r), \quad \text{for all } \varepsilon \in \mathcal{O}_K^\times,$$

$$(3.6) \quad c(n, r) \ll_\phi \mathbf{N}(4nm - r^2)^M$$

for an integer M .

Lemma 3.1. (1) *Condition (3.4) implies that we can decompose $\phi(\tau, z)$ as in (2.3).*

(2) *Conditions (3.4) and (3.5) imply by the definition of T_m that*

$$c_r(N) := c\left(\frac{N + r^2}{4m}, r\right), \quad N \in \mathfrak{d}_K^{-2},$$

is well defined on $r \in \mathfrak{d}_K^{-1}/(2m\mathcal{O}_K)$, where we put $\mathfrak{d}_K^{-2} := \mathfrak{d}_K^{-1} \cdot \mathfrak{d}_K^{-1}$.

(3) *ϕ is a Hilbert-Jacobi cusp form if and only if (3.3), (3.4), (3.5) and (3.6) hold, and if ϕ satisfies the transformation law*

$$(3.7) \quad \phi\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = \mathbf{N}(\tau)^k e\left(\frac{mz^2}{\tau}\right) \phi(\tau, z).$$

(4) *From Corollary 2.2, we see that a function ϕ satisfying (3.3), (3.4), (3.5) and (3.6) is a Hilbert-Jacobi cusp form if and only if (2.6) is satisfied for all $r \in (\mathfrak{d}_K^{-1}/2m\mathcal{O}_K)$.*

Proof. These are straightforward. We omit the proof of this Lemma. \square

Let us now define the Dirichlet series needed for Theorem 1.1.

Definition 3.2. For a function ϕ satisfying (3.3), (3.4), (3.5) and (3.6), $r \in \mathfrak{d}_K^{-1}/(2m\mathcal{O}_K)$, and ν satisfying (3.1) and (3.2), we define

$$(3.8) \quad L(s, \phi, r, \chi_{m,\nu}) := \sum_{\alpha \in \mathfrak{d}_K^{-2}/T_m^2} \chi_{m,\nu}(\alpha) \cdot c_r(\alpha) \cdot \mathbf{N}(\alpha)^{-s},$$

$$(3.9) \quad \mathcal{L}(s, \phi, r, \chi_{m,\nu}) := 2^{gs} \pi^{-gs} \prod_{j=1}^g \Gamma(s - \nu_j) \mathbf{N}(m)^s \\ \times \prod_{j=1}^g \left(m^{(j)}\right)^{-\nu_j} L(s, \phi, r, \chi_{m,\nu}).$$

Due to (3.6), the series $L(s, \phi, r, \chi_{m,\nu})$ is absolutely convergent for $\sigma = \text{Re}(s) > M + 1$.

We have the following lemma.

Lemma 3.3. For $\sigma > M + 1$, we have the identity

$$(3.10) \quad \mathcal{L}(s, \phi, r, \chi_{m,\nu}) = \int_{T_m^2 \setminus \mathbf{R}_+^g} f_r(iy) \mathbf{N}(y^{s-\nu}) \frac{dy}{\mathbf{N}(y)},$$

where $f_r(\tau)$ is the form defined in (2.4) with Fourier coefficients $c(n, r)$.

Proof. This can be directly calculated by using the Fourier expansion of $f_r(iy)$ and by using the relation $\mathbf{N}(u_l^\nu) = \mathbf{N}(\varepsilon_l)^k$ for $l = 1, \dots, g - 1$, where u_l and ε_l are defined in the beginning of this section. We leave the details to the reader, see also [3, page 87]. \square

4. Proof of Theorem 1.1. Theorem 1.1 follows directly from the two lemmas proven in this section.

Lemma 4.1. If ϕ is a Hilbert-Jacobi cusp form, $r \in \mathfrak{d}_K^{-1}/(2m\mathcal{O}_K)$ and ν satisfies (3.1) and (3.2), then the functions $\mathcal{L}(s, \phi, r, \chi_{m,\nu})$ have

analytic continuations to the whole complex plane. They are of rapid decay and satisfy the functional equations

$$(4.1) \quad \mathcal{L}(s, \phi, r, \chi_{m, \nu}) = \frac{1}{\sqrt{D_K}} i^{-kg} \mathbf{N}(2m)^{-1/2} \times \sum_{\mu \in (\mathfrak{d}_K^{-1}/2m\mathcal{O}_K)} e_{2m}(-\mu r) \mathcal{L}(k - s - 1/2, \phi, \mu, \chi_{m, -\nu}).$$

Proof. To prove the analytic continuation of $\mathcal{L}(s, \phi, r, \chi_{m, \nu})$, we show that the righthand side of (3.10) is analytic for all s . For this, we separate the integral into a part with $\mathbf{N}(y) \geq 1$ and a part with $\mathbf{N}(y) \leq 1$. Using the transformation law of f_r , one can see that it is enough to consider the part with $\mathbf{N}(y) \geq 1$. To estimate this, we use the variables $y_0 \in \mathbf{R}_+$ and $t = (t_1, \dots, t_{g-1}) \in \mathbf{R}^{g-1}$, where

$$y_j := y_0 \cdot e^{\sum_{i=1}^{g-1} t_i \log(u_i^{(j)})}.$$

Then a fundamental domain of $T_m^2 \setminus \mathbf{R}_+^g$ is given by the inequalities $y_0 > 0$ and $0 \leq t_l < 1, l = 1, \dots, g - 1$, and the part with $\mathbf{N}(y) \geq 1$ is given by $y_0 \geq 1$. The analyticity now follows if we use Lemma 2.3, since for $c > 0$ and $\sigma \in \mathbf{R}$ arbitrary the integral $\int_1^\infty e^{-cy} y^\sigma dy$ is convergent. The boundedness of $\mathcal{L}(s, \phi, r, \chi_{m, \nu})$ in every vertical strip also follows from this convergence.

Moreover, by using the transformation law of f_r and Lemma 3.3, equation (4.1) follows since $1/y$ runs through $T_m^2 \setminus \mathbf{R}_+^g$ if y does. \square

Lemma 4.2. *Assume that ϕ is a function satisfying (3.3), (3.4), (3.5) and (3.6), and that for all $r \in \mathfrak{d}_K^{-1}/(2m\mathcal{O}_K)$ and for all ν satisfying (3.1) and (3.2) the series $\mathcal{L}(s, \phi, r, \chi_m)$ have analytic continuations, satisfy (4.1) and are of rapid decay. Then ϕ is a Hilbert-Jacobi cusp form of weight k and of index m .*

Proof. By analytic continuation it is enough to show (2.2) for $\tau = iy$. We parametrize the integrals as before and use the Mellin inversion formula to get, for σ sufficiently large

$$(4.2) \quad \int_{[0,1]^{g-1}} f_r(iy_0 e^{tR}) e^{-\nu tR} dt = \frac{1}{2gR_m \pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \mathcal{L}(s/g, \phi, r, \chi_{m, \nu}) y_0^{-s} ds,$$

where

$$R_m := \det((\log(u_l^{(j)}))_{l,j=1,\dots,g-1})$$

$$f_r(iy_0 \cdot e^{tR}) := f_r\left(iy_0 e^{\sum_{l=1}^{g-1} t_l \log(u_l^{(1)})}, \dots, iy_0 e^{\sum_{l=1}^{g-1} t_l \log(u_l^{(g)})}\right),$$

$$(4.3) \quad e^{-\nu tR} := \prod_{j=1}^g \prod_{l=1}^{g-1} e^{-\nu_j t_l \log(u_l^{(j)})} = \prod_{l=1}^{g-1} e^{-2\pi i(N_l + (1/2)\delta_l)t_l},$$

where N_l and δ_l appeared in (3.2). Applying (4.1) and making the substitution $s \rightarrow g(k - 1/2 - s)$ gives that the righthand side of (4.2) equals

$$\begin{aligned} & \frac{1}{\sqrt{D_K}} i^{-kg} \mathbf{N}(2m)^{-1/2} \sum_{\mu \in (\mathfrak{d}_K^{-1}/2m\mathcal{O}_K)} e_{2m}(-\mu r) y_0^{-g(k-1/2)} \\ & \times \frac{1}{2gR_m \pi i} \int_{g(k-1/2)-\sigma-i\infty}^{g(k-1/2)-\sigma+i\infty} \mathcal{L}(s/g, \phi, \mu, \chi_{m,\nu}) y_0^s ds. \end{aligned}$$

If $\text{Re}(s) > M + 1$, the series $L(s, \phi, r, \chi_{m,\nu})$ is absolutely convergent, and the series $\mathcal{L}(s, \phi, r, \chi_{m,\nu})$ is of rapid decay for $|\text{Im}(s)| \rightarrow \infty$. Also, $\mathcal{L}(s, \phi, r, \chi_{m,-\nu})$ is bounded in every vertical strip and has a functional equation. By using the Phragmén-Lindelöf principle, we can conclude that $\mathcal{L}(s, \phi, r, \chi_{m,-\nu})$ is of uniformly rapid decay for $|\text{Im}(s)| \rightarrow \infty$ in every vertical strip. Hence, we use Cauchy's theorem and shift the path of integration to the line $\text{Re}(s) = \sigma$. Thus, the lefthand side of (4.2) equals

$$\begin{aligned} & \frac{1}{\sqrt{D_K}} i^{-kg} \mathbf{N}(2m)^{-1/2} \sum_{\mu \in (\mathfrak{d}_K^{-1}/2m\mathcal{O}_K)} e_{2m}(-\mu r) y_0^{-g(k-1/2)} \\ & \times \frac{1}{2gR_m \pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{L}(s/g, \phi, \mu, \chi_{m,-\nu}) y_0^s ds. \end{aligned}$$

But the latter integral equals

$$2gR_m \pi i \int_{[0,1]^{g-1}} f_\mu(iy_0^{-1} \cdot e^{-tR}) e^{-\nu tR} dt.$$

Thus,

$$\begin{aligned}
 (4.4) \quad & \int_{[0,1]^{g-1}} f_r(iy_0 \cdot e^{tR}) e^{-\nu tR} dt \\
 &= \frac{1}{\sqrt{D_K}} i^{-kg} \mathbf{N}(2m)^{-1/2} \sum_{\mu \in (\mathfrak{d}_K^{-1}/2m\mathcal{O}_K)} e_{2m}(-\mu r) y_0^{-g(k-1/2)} \\
 & \quad \times \int_{[0,1]^{g-1}} f_\mu(iy_0^{-1} \cdot e^{-tR}) e^{-\nu tR} dt.
 \end{aligned}$$

We now let

$$\begin{aligned}
 g_r(t) &:= f_r(iy_0 \cdot e^{tR}) - \frac{1}{\sqrt{D_K}} i^{-kg} \mathbf{N}(2m)^{-1/2} \\
 & \quad \times \sum_{\mu \in (\mathfrak{d}_K^{-1}/2m\mathcal{O}_K)} e_{2m}(-\mu r) y_0^{-g(k-1/2)} f_\mu(iy_0^{-1} \cdot e^{-tR}).
 \end{aligned}$$

To prove the lemma, it suffices to show that $g_r(t)$ is identically zero. But this follows since the function

$$\hat{g}_r(t) := g_r(t) \prod_{l=1}^{g-1} e^{-\pi i \delta_l t_l}$$

has period 1 in every component of t and all (N_1, \dots, N_{g-1}) th Fourier coefficients of $\hat{g}_r(t)$ are 0 due to (4.3) and (4.4). \square

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