Didactical designs for students' proportional reasoning: An "open approach" lesson and a "fundamental situation"*

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Abstract.
In this paper we analyse and compare two didactical designs for introducing primary school pupils to proportional reasoning in the context of plane polygons. One of them is well documented in the literature; the other one is based on our own data and is accordingly presented and discussed in more detail here. The two designs come from different cultural and intellectual environments: lesson study in Japan (implicitly based on the "open approach method") and "didactical engineering" in France (based on the theory of didactical situations). The general aim of our paper is to compare these two environments and their approaches to didactical design, basing our discussion on the concrete designs mentioned above. Clear differences among them will be presented, while we also identify links which hold potential for integrating research and practice.

Key words.
Proportionality, Similarity, Didactical design, Open Approach Method, Theory of Didactical Situations, Lesson Study, Didactical Engineering**.**

Introduction

We consider that there are at least two grand types of approach to improving mathematics teaching and learning (with some room for intermediate approaches).

On the one hand, improvement is envisaged by means of understanding teaching and learning processes from a research perspective, looking at teaching as a design object (see e.g. Artigue, 2009). To do systematic design, one needs explicit and precise models of teaching and learning processes; different models are provided by different theoretical frameworks (see e.g. Prediger et al. (Eds.), 2008). The last decades have seen a proliferation of new theories within mathematics education research, and this leads to a concern about establishing more links between the models and tools they provide, in particular for design purposes (cf. Artigue et al., 2006).

On the other hand, teachers' ordinary practice inside or outside of the real classroom has led to more systematic ways of improving mathematics teaching and learning, rooted within a cultural context. These practices are not necessarily based on an explicit theoretical framework, but relating them to research perspectives might be an important strategy to close the "gap" between practice and research which is an increasingly recognised challenge (cf. e.g. Even and Ball (Eds.), 2003; Heid et al., 2006).

The two types of approach address the concrete purpose of designing a lesson in different ways. In the first type, a theoretical framework allows researchers or educators to design a lesson based on

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the analysis of mathematical tasks according to the theoretical principles proposed in a theory. In the second type, a lesson is designed based on the analysis of mathematical tasks according to the teacher's experience of the reality in the classroom. Each approach takes into account different dimensions of mathematics education, and each of them could support the other, in order to develop a body of scientific knowledge in our domain on the one hand and to develop the practice of teaching mathematics in a specific culture on the other.

In this paper, we are interested in a comparative analysis of lesson designs that are outcomes of two prominent approaches of the two types, rooted in different cultural and intellectual environments:

- *didactical engineering* in France (see e.g. Artigue, 1992); this approach is explicitly based on a theory, normally the theory of didactical situations (cf. Brousseau, 1997 and Sec. 2.4).
- *lesson study* in Japan, (cf. Isoda et al., 2007 and Sec. 2.1); the lessons usually involve "mathematical problem solving" activities (cf. Hino, 2007), quite often realised with the "open-approach method" (cf. Nohda 2000; Becker and Shimada, 1997).

Our main objective is to unfold similarities and differences between these two approaches in order to clarify what each provides. To accomplish this objective, we study two specific lesson designs for students' first encounters with proportionality: how do they affect students' and teachers' work in the classroom? What principles do they reflect? More precisely, in terms of the theory of didactical situations, we consider two *didactical milieus* for early proportional reasoning, and the associated or resulting *didactical contracts* that can be inferred from classroom observation (cf. also Hersant and Perrin-Glorian, 2005). We then study the didactical principles and processes behind the design of these milieus in order to identify links and differences.

The structure of this paper is as follows: in Section 1, we provide a short discussion of the mathematical, historical and didactical background for the theme of proportionality. In Section 2, we introduce our empirical material and context: the design and realisation of an inquiry oriented lesson that forms part of an introduction to proportionality, given at a primary school in Japan, in the context of a lesson study. In Section 3, we analyse the classroom situation more deeply, using the theory of didactical situations (as an analytic framework, rather than as a design tool). In Section 4 we compare our observations and results from the Japanese setting to a famous lesson design by Brousseau, and we discuss how to relate the two approaches to lesson design as well as the two underlying theories (open approach methods and the theory of didactical situations, respectively).

Notice that the structure of this paper is "inductive", in the sense that we first study some concrete problems and cases of lesson design, and then discuss wider perspectives and principles. This is consistent with the historical development and basic ideas of the above theoretical frameworks.

1 Preliminaries on proportionality

Proportional reasoning is an important theme in mathematics education practice and research, particularly at the elementary school level in any country (see e.g. Tourniaire and Pulos, 1985). It occurs in a variety of mathematical and real world contexts, and it is crucial in early learning related to numbers and geometry. Any primary school mathematics teacher will make or accept choices on how to introduce students to the idea of proportionality and, more broadly, organise the students' first work with proportions and proportional reasoning in various contexts. In this section, we recall some basic mathematical, historical and didactical points about the somewhat heterogeneous notion of “proportionality”. These points will be of importance in our analysis of didactical designs and students’ reasoning in this setting.

1.1 Two definitions of proportionality

A common way to present proportionality in mathematics is as a special kind of dependence between two variables x and y, namely the situation where we have $y = kx$ for a fixed constant k. The “variables” here are normally understood as real variables, i.e. they could take real numbers as values. Whether we talk of functions or variables, we often think about a fixed relationship between an “input” and an “output”; so we will call this a *dynamic* definition of proportionality.

We may also think of proportionality in terms of subsets of $\mathbb{R}^n$ of form $\hat{a} = \{(sa_1, \ldots, sa_n) : s \in \mathbb{R}\}$, where $a = (a_1,\ldots,a_n)$ is fixed. If $a, b \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ we say that a is proportional to b (or we simply write $a \sim b$) if $b \in \hat{a}$. We call this a *static* definition of proportionality since it defines what it means for two fixed n-tuples of real numbers to be proportional. In the case $n = 2$, it relates to the dynamic definition as follows: the “slope” $k = a_2/a_1$ and the “direction” (a_1, a_2) correspond in the two settings.

The static definition is, in a way, more general than the dynamic definition, since it allows us to define proportionality of n-tuples of real numbers, rather than just proportionality of pairs. This more general case is relevant e.g. to similarity of polygons, and many other applications.

1.2 Proportionality and similarity in Euclid’s Elements.

The first systematic treatment of proportionality is found in Euclid’s *Elements* (cf. Joyce, 1997), where part of the background is in fact implicit: the discovery of simple figures with incommensurable parts. Even if the general theory of proportionality, as found in book V of Euclid’s Elements, may seem somewhat obsolete in modern mathematics – or in contemporary school curricula with their pragmatic stance on irrational numbers – some of the ideas are relevant for us.

Euclid’s definition of proportionality (Elements, book V, definition 5 and 6) is static in nature, and deals with pairs of “magnitudes” rather than numbers. A magnitude could be a length – like the diagonal of a square – which, for the Greeks could not be measured relative to the side, but nevertheless multiplied in the geometric sense. Two pairs of magnitudes (a, b) and (c, d) are then said to be proportional if for any natural numbers m, n we have the following implications (in modern notation):

$$ma = nc \Rightarrow mb = nd,\ ma > nc \Rightarrow mb > nd,\ ma < nc \Rightarrow mb < nd.$$

If we consider this definition on $\mathbb{R}^2 \setminus \{\mathbf{0}\}$, it can be shown that it is in fact equivalent to the static definition of proportionality given above (see Joyce, 1997); the essential idea is that real numbers may be approximated arbitrarily well by rational numbers. The definition by three implications may seem, at a first encounter, unpractical and complicated; notice, however, that to show *non-proportionality* of (a, b) and (c, d) it does not ask for more than the demonstration that one single implication is violated for one pair of integers.

In book VI, Euclid goes on to the geometric notion of similarity. Two polygons are called similar if they *have their angles severally equal and the sides about the equal angles proportional* (definition

1, taken from Joyce, 1997). Notice that the definition comprises two parts: the angles should be the same (implicitly, there should be the same number of angles, and they should be ordered in such a way that angles with same place in this order are equal); and sides around angles with the same place in the order should be proportional. In theorems 5 and 6, Euclid shows that for triangles, each of the two conditions implies the other, so that one of them suffices to conclude similarity. This is one of the most basics facts of geometry – and the proof is not simple, even when using modern terms. Notice also that for polygons with more than three sides, both parts of the definition are needed. However, the simpler notion for triangles can be made useful for higher polygons: in proposition 20, Euclid shows that similar polygons can be divided into similar triangles (*triangulation*, in modern terms), so that if this cannot be done, the polygons are not similar.

1.3 Proportionality in mathematics education

The theme of proportionality appears in various ways in the mathematics education literature. It is impossible to review this literature in a single paper (see Tourniaire and Pulos, 1985, for an early attempt). A large part of the literature deals with the difficulties of young students to reason "proportionally" or "linearly", for instance as an alternative to "additive reasoning" (we return to this problem in the sequel), while another part of the literature shows how this problem may be so well overcome that all variable relations are seen as linear (see e.g. de Bock et al., 1998).

In primary school curricula, "proportional reasoning" occurs often through simple numeric word problems in which three numbers a, b, and c are given, and a fourth one d must be found – in a situation which implies that (a, b) and either (c, d) or (d, c) are proportional in the sense of the static definition (see for instance de Bock et al., 1998; Tourniaire and Pulon, 1985). Very often, *units* and *rates* are involved, which adds complexity and familiarity at the same time (e.g. for magnitudes related to time, distance, cost and so on). Another important domain in school mathematics is that of proportional reasoning related to similar plane figures (such as maps of different *scales*), which may cause problems even for student teachers (Winsløw and Durand-Guerrier, 2007).

The dynamic definition of proportionality tends to show up later on, often as a shorthand for the relation between a potentially infinite collection (x, y) of numbers that are all mutually proportional (such as distances y and corresponding travel times x, under the assumption of constant speed). In this sense the dynamic definition appears, from a didactical viewpoint, to be more general and advanced, especially when presented in an algebraic setting (involving variables or even functions). While algebraic treatment (cf. Adijage and Pluvinage, 2007, p. 153) may help to trivialise notions and problems related to proportionality for secondary school students, one normally has to do without it in first approaches at primary school. And this adds to the challenge of constructing "first encounters" with proportional reasoning at this level. The literature also contains important concrete ideas for doing so. We return, in Section 4, to one famous example; in Section 3, we analyse a new one.

2 Context and framework for analysis of the Japanese lesson design

In this section, we briefly explain the context and nature of our data which constitute the basis for our research on the themes of proportionality and lesson design as reported on in the sequel. We also introduce the basic elements of the theory of didactical situations that we will use to analyse the data.

2.1 A short note on lesson study

Lesson study is a format for teacher led “action research” (see e.g. Krainer, 2006) focusing on the development of one lesson with specific aims and relations to the curriculum and a surrounding sequence of lessons. This format, of Japanese origin, is extensively documented in the literature (see e.g. Fernandez and Yoshida, 2004; Isoda et al., 2007). We only give a short introduction here.

According to Fernandez and Yoshida (2004, p. 15), the “vast majority of elementary schools and many middle schools conduct *konaikenshu*”, which means literally “training/study inside the school”. Lesson study is an important and common form of *konaikenshu*. To fully explain the mechanisms and conditions of lesson study in its present form, one needs to provide a much broader picture of the functioning of the Japanese system of education than we can afford here. But briefly speaking, the Japanese mathematics teachers have the means and habit of studying together in teams at each elementary school, with hours and workspace set aside for such work.

The heart of a lesson study is the development and testing of a *research lesson*, based on careful study of available resources and constraints for this lesson. It evolves around a central document: the *lesson plan*, which describes in detail the aims, background and workings of the lesson. It is a collective product of the group, and it is typically tested in different classes and with different team members as teachers, while the rest observe and take notes. Meetings before and after these test lessons serve to focus and discuss the observations, and to revise the lesson plan accordingly. The product, in the form of a well tested lesson plan, may be shared with colleagues, even at other schools, through informal meetings, teacher congresses, and commercial publications.

2.2 The Japanese curriculum on proportionality

In English, there seems to be a large variation in how words like *fraction*, *ratio* and *proportion* are used and related (cf. Clark et al., 2003). For our analysis of the lesson, and to describe the corresponding curriculum, we need to introduce some similar Japanese words, and then use them to define the English words, as they will be used in this paper. Note that the translations of these words are different among some Japanese authors writing in English. In this article, we follow the translations of JSME (2000).

The notion of *wariai* (which we choose to call *ratio* in English) is introduced in the 5th grade of Japanese elementary school (10-11 years old students), in the context of *ryou* (magnitudes) and *tan-i* (units). In a common text book for this level we read (in our translation):

> We call "ratio" the number which expresses a magnitude compared to a unit magnitude. The ratio can be obtained by the following formula: "Ratio = Compared magnitude ÷ unit magnitude". (*New Mathematics* 5 vol. 2, Tokyo Shoseki, 2005, p. 41)

Examples given in this textbook include reductions of a price (1400 yen → 1000 yen: 5/7), the number of won games in relation to the total number of games, speed (so one may have a ratio of magnitudes of different kinds). The notion of *percentage* is introduced at the same time, as a way to express a ratio of magnitudes of the same kind. Notice that the *ratio* may be described in two ways, as a fraction (number) or as a percentage (different unit).

The notion of *proportion* (*hi*, in Japanese) is introduced in 6th grade (11-12 years old students) as another way to express a ratio, which does not imply division:

> The ratio of 2 to 3 is often expressed with the notion ":", as "2 : 3", which reads "2 to 3". The ratio expressed like that is called the *proportion*. (…) The proportion is a way to express a ratio by two numbers. The ratios learned in 5th grade use a method to express the ratio as a single number". (*New Mathematics* 6 vol. 2, Tokyo Shoseki, 2005, p. 34f)

We may interpret these explanations as saying that the value resulting from division (a fraction), as well as a proportion, are also two ways to express a ratio between magnitudes of the same kind.

At the same time, the idea of proportionality is introduced through the notion of *equivalent proportions*. The national curriculum for 6th grade (JSME, 2000, pp. 17-20) stipulates that:

> [Contents, Mathematical relationships] Children should understand the meaning of proportion in simple cases
>
> [Points for consideration when dealing with contents] … by investigating quantitative relationships in concrete situations, treatment should only extend to understanding equivalent proportions; the value of proportions should not be handled.

The last remark means that proportional relationships are studied as expressions like 2:3 and 4:5, and that students should investigate whether they are "equivalent" *without* using the value (fraction) to compare them; or at least that they should chiefly occur in situations where the use of fractions is not the expected or natural thing to turn to. Notice that the notion of equivalence of proportions implied for 6th grade is very close to the static definition of proportionality introduced here in Sec. 1.1, with only a difference in notation (namely, $a:b$ instead of (a, b) is used to express an ordered pair of numbers). On the other hand, the similarity of polygons that could relate to the static definition of proportionality does not appear in the curriculum until 9th grade (14-15 years old students) where the similar triangles are taught (ibid., p. 26).

2.3 Theory of didactical situations

As mentioned above, the theory of didactical situations (TDS) is adopted as an analytic tool for the analysis of our data. This framework is well documented in English (e.g., Brousseau, 1997; Warfield, 2007). Before presenting our analysis, we briefly present the elements of TDS that we use.

According to TDS, students' learning of a specific piece of mathematical knowledge intended to be taught is modelled with the notion of *didactical situation* in which interactions occur between three elements: *students*, *teacher* and *milieu*. The significant learning happens when the student interacts with an objective milieu as if the teacher is absent for the student (*adidactical situation*; *adidactical milieu*), that is to say, when the students' action does not depend on the teacher. And TDS postulates that "each item of knowledge can be characterized by a (or some) adidactical situation(s) which preserve(s) meaning" (Brousseau, 1997, p. 30). Such a situation is called a *fundamental situation*.

The relationship between student and teacher in a given situation is described by the notion of *didactical contract* which is one of key notions in TDS. It is understood as the system of mutual expectations, most of the time implicit, between the teacher and the student with respect to the mathematical knowledge at stake. It is this contract that determines the responsibility of students and teacher in the situation, and the contract regulates their interaction. Throughout the process of teaching and learning, the didactical contract is not necessarily stable, but can be modified by the teacher (*devolution* of the milieu, and *institutionalization* of acquired or established knowledge).

This notion of didactical contract has been further developed since it was initially introduced, in order to characterize different kinds of contract. In this paper, we refer to three different levels in the structure of the didactical contract which were proposed by Hersant & Perrin-Glorian (2005): *macro-contract*, *meso-contract* and *micro-contract*.

> "The macro-contract is mainly concerned with the teaching objective, the meso-contract – with the realization of an activity, e.g. the resolution of an exercise. The micro-contract corresponds to an episode focused on a unit of mathematical content, e.g. a concrete question in an exercise." (ibid., p. 119)

These notions will help us in analysing certain crucial differences between the "open approach" based design and a design based on fundamental situations.

2.4 Our data and methods

The lesson which we study in this paper resulted from a lesson study at an ordinary public primary school in Tsukuba, Japan, in 2006/2007. The team leader, Mr Chikara Kobayashi, and his colleagues, generously provided us with data from this lesson study, in particular a video recording of the lesson, the lesson plan as well as videos and notes from discussions in the group. One can say that this group is quite representative of lesson study groups in Japan.

These data were first gathered to prepare a presentation of the format of lesson study at a conference in France (Miyakawa and Winsløw, 2008). The video data were transcribed (naturally, in Japanese) and all data were translated into French (which is the main common language of the authors). As we worked on the presentation, we began to notice various connections and subtle differences with the famous puzzle situation by G. Brousseau (described here in Sec. 4.1). Also, the presentation of lesson study itself led us to consider more general differences in the ways lesson structure is conceived and described in the French and in the Japanese context.

In the present study, we have focused on the transcribed video data from one test of the lesson for 6^{th} graders (11-12 years old students), as well as on the corresponding lesson plan. Our analysis of the lesson began with identifying the *main phases* in the lesson (cf. Sec. 3.2) as it is realised in the classroom. We also identified the *main strategies* which the students develop to cope with the mathematical problem they are presented within the lesson (outlined in Sec. 3.3). All of these can be objectively associated to the video transcript of the lesson. Finally, in Section 3.4 we argue that in order to account for the development of the students' strategies in the lesson, it is helpful to consider the *didactical contract* at various levels (cf. Hersant and Perrin-Glorian, 2005, sec. 1.2.4). To extract the main elements of the contract, we obviously have to interpret our data more globally, as didactical contracts are largely implicit (cf. Brousseau, 1997, p. 32); but we did so with maximal backing in the data, including also the recording of the teacher team's discussion before the lesson.

Therefore our main data are related to a single lesson, not some consecutives lessons over time. This unit of analysis for our research is determined by the institutional time constraint. However, this definition of the unit is relevant for our analysis of milieu and didactical contract, as long as a lesson contains, in terms of the theory of didactical situations, some particular situations where the notion of proportionality is implicitly or explicitly dealt with and eventually put into shape (situation of action, situation of formulation, and situation of validation), that is where the didactical milieus for teaching proportionality are at stake. This means that using a single lesson as the unit of analysis is theoretically justified from the point of view of TDS. But notice that this is also the unit of analysis in lesson study (Sec. 2.1), and due to this we have significant data not only from the lesson itself but also from its planning and evaluation, that is, from the design process involved in a lesson study; in other words, we can study not only the lesson itself, but also explicit aspects of its design.

3 Analysis of the Japanese lesson "Reflect on the meaning of 'same form'"

Here is a short and somewhat naïve outline of the lesson as it can be observed from the video data. First, the teacher chats with the students about a group picture of the class taken some months ago, which he has put up on the board in three different sizes. After about 6 minutes of small talk – that appears to amuse most of the students – the teacher asks them to take out their notebooks and draw a square of side 3 cm, and after that a square of side 5 cm. They are asked if the two are of the same form, or not; everyone agree that they are of the same form. Then the teacher asks the class to draw two rectangles in their notebooks, one of size 3cm × 5cm, and then one of size 5cm × 7cm. He also asks the students to write, in the notebook, their "immediate impression" as to whether the two rectangles are of the same form. After a few students have aired their contrasting opinions in the class, the students are asked to work (for about 5 minutes) to justify their claim. Then we are about half through the lesson. In the last part, the teacher successively asks different students to explain their work to the whole class; first, some students who think the two rectangles are of the same form, then some students who think they are not. The discussion is lively, and different reasons are given for both conjectures. The lesson ends with no final conclusion explicitly favoured by the teacher, however implicitly the conjecture of “different form” is recognised to prevail, because the teacher gives the following assignment for homework: “if you think the rectangles are of different form, construct another rectangle of the same form as the smaller one”.

3.1 Aims and preparation of the lesson

The lesson plan describes in detail the aims and proceedings of the lesson. It is the second lesson in a sequence of seven lessons, entitled “Reflect on expressions of ratio” and outlined as follows:

1. Reflect on the meaning of “same form” [for different size drawings]
2. *Reflect on the meaning of “same form” from the points of view of proportions of two magnitudes* [this is the lesson we consider, and the *research lesson* in this lesson study]
3. Understand the meaning of the expression of proportion and compute its value
4. Understand the meaning and expression of equal proportions
5. Understand how to determine if two proportions are equal
6. Using different methods to find the value of proportions
7. Reflect on how to shift between expressions of ratio (decimal numbers, fractions, proportion)

As one sees, the first lesson is inserted to prepare the second one, while the notion of “same form” is merely a key example for the work with ratio, in this sequence.

The plan begins by quoting the aims from the teacher’s textbook; they are very close to the official aims for 6th grade cited in Section 2.3. In the plan, the team emphasises that in an expression like “ratio of 2 to 3”, the two numbers enjoy an equal status, as opposed to a fraction or percentage where one of the two numbers is taken as a unit for the comparison. The whole teaching sequence aims for the students to reflect on the proportional expression of type “2:3”, and to develop ways to understand and determine the equivalence of two ratios expressed in this way (proportionality). The description of the aims seems more focused on the process of students’ thinking than the official programme, which emphasises the overall target knowledge for students.

One of the most interesting sections of the lesson plan is entitled “The reality of the students”; it discusses the particular resources and constraints for the students in this class with respect to the subject of ratio, as they have been observed by the teachers in previous teaching. In particular, they have conducted a small “test” on ratio expressed as fractions. The results suggest that most students in the class are insecure about how to distinguish a/b and b/a in a concrete setting. The team therefore chose to construct a lesson that provides a strong concrete experience of the meaning of ratio, using proportions. Japanese textbooks often exemplify proportions by different solutions of vinegar in oil, or syrup in water, but the team had found this to be partially unsuccessful due to the insufficient difference in taste, particularly for detecting subtle differences (such as 2 versus 3 teaspoons of vinegar in a dressing). So they decided to use the concept of “same form” in the context of rectangles. That is to say, they chose a milieu whose potential feedback and information could be based on visual (rather than taste) perception.

During a discussion prior to the lesson – where all the main points just mentioned are already decided – the teachers of the lesson study team discuss the details of the research lesson. A main point is that, even after the first lesson, the students may have different ideas about the meaning of “same form”; for instance, they might say that any two rectangles have the same form, being rectangles. However, they conclude that if the problem for the students is to determine whether two given rectangles are of the same form or not, more subtle ideas will be dominant and will have to be negotiated. We note that, implicitly, this can be interpreted as the teachers' expectation for the didactical contract: the students will know that the answer cannot simply be, for instance, "yes, they are of the same form because they are both rectangles". The teachers expect the students to expect more resistance in the problem situation proposed.

Another main point in the preparatory discussion observed is a specific detail of the milieu to be implemented in the lesson, namely the choice of the actual proportions of the figures to compare. At this point, the question asked to the students (whether two rectangles are of the same form) and the way to obtain two rectangles (an original rectangle is given and another one is formed by prolonging each side of the original by 2cm) have been already decided. For the original rectangle, a simple proportion, 3:6, is discarded because the corresponding rectangle is a double square, which may be easily recognised as different from the enlarged rectangle whose ratio of sides is 5:8, without considering the proportion itself. They therefore settle on the ratios 3:5 and 5:7. Then they discuss the material milieu (how to materially present the students with the corresponding rectangles). Should these figures simply be given to the students on a sheet of paper? The teachers agree that the students may then try to superimpose the figures along the common side, instead of focusing on the proportions. Therefore, the students should themselves construct the two rectangles successively in their notebooks, noting explicitly that 5×7 is obtained from the other by prolonging each side by 2. To prepare this idea, as well as the problem of "same form", the idea arises to begin with proposing the easy case of two squares (3×3 and 5×5), which should also be drawn successively by students. We may see here that an objective milieu is set up so that it gives visual information and feedback about "same form" that require students to consider spontaneously the proportion (as an adidactical milieu for proportional thinking). However, notice that this milieu might not be adidactical in the sense of allowing students to see the correctness of their answer by themselves, and in fact lessons based on the open approach do not emphasise this feature of the milieu.

The structure of the lesson thus decided is recognised in the four main phases drawn up in the section of the lesson plan entitled "Teaching of this lesson":

1. Grasp the task of this lesson (construction of squares, see they are of the same form; construction of rectangles, then see whether they are of the same form)
2. Individual work based on immediate conjecture and subsequent reasoning; teacher circles silently in the class to identify students with different conjectures and justifications
3. Discussion where the students listen to understand other students' explanations, and defend their own
4. Return to individual work [in the taught lesson, this becomes homework due to time constraints, as mentioned above].

This is, of course, a central part of the lesson plan, as it describes the research lesson in concrete detail; along with the above phases, it contains numerous points, such as the importance that students formulate their reasoning in their notebook as clearly as possible during the individual work. The plan is arranged in a table with the above phases as rows, and the following columns: "Contents and activity"; "Form of the activity" (all together or individual); "Support and evaluation".

The remaining part of the preparatory discussion focuses on the perceptual and reasoned ideas which are likely to come up among the students (in some cases, the teachers even predict what an individual student may think). Some potential hypotheses and strategies are mentioned in the lesson plan:

a. draw the diagonals and measure the angles → different form
b. superimpose the two figures and find the same difference along the sides → same form
c. superimpose the two figures with diagonals drawn, not superposable → different form [one teacher bets a steak restaurant visit on his conjecture that this strategy will arise among students!]
d. perceptual difference: the small rectangle is closer to a square → different form
e. value of proportion (1.7 vs. 1.4) → different form

During the lesson, the teacher should try to help the students formulate clearly their hypothesis and justifications, as many as possible. Here we see a characteristic feature of "open approach" situations: multiple strategies are expected and acceptable, and thus every student can be engaged. The openness thus resides in the process, not in the answer (same form or not). The students' activity is open and they can propose various arguments supporting their answer.

3.2 Lesson structure

A lesson may be everything from a lecture, where the teacher talks while the students listen, to a period of group work or individual seatwork, where the teacher is available to answer questions. But most lessons are composed of distinct phases with different work modes and purposes; the time wise organisation of these phases (which may be characterised according to different models) is what we mean, loosely speaking, by the term "lesson structure". An important synthetic outcome of the TIMSS video studies, in which hundreds of lessons in different countries were videotaped and analysed, was that lesson structure is often surprisingly similar from school to school *within* a given cultural context, and not less surprisingly *different* between schools in different cultural contexts:

> The systems of teaching within each country look similar from lesson to lesson. At least, there are certain recurring features that typify many of the lessons within a country and distinguish lessons among countries. There recurring features, or *patterns*, define different parts of a lesson and the way the parts are sequences. (Stigler and Hiebert, 1999, p. 77f)

In particular, Shimizu (1999) and Stigler and Hiebert (1999, pp. 79-80) identified essentially the following structure for a typical lesson in Japan:

- *Hatsumon* ("questioning" in Japanese): The teacher introduces an "open problem";
- *Kikan-shido* ("instruction at the desk"): The students work on the problem – typically individually – while the teacher circulate to observe their work and identify different approaches, and clarify the problem if needed;
- *Takuto* ("orchestra conducting"): The teacher asks students to successively present their solutions or ideas for the *hatsumon* to the whole class, making sure that different ideas come out and are understood by all;
- *Neriage* ("elaboration", often integrated into *Takuto*): discussion of the validity and pertinence of the proposed ideas, mainly based on students' contribution, but sometimes involving the teachers' evaluation (but still within a whole class instruction mode);
- *Matome* ("summing up"): the teacher recalls the main points of the lesson, sometimes pointing out or even reformulating the best or new methods found.

We clearly recognize most of these phases in the four points for "Teaching of this lesson" in the lesson plan, which were explained in the previous section. In particular we emphasise the central role of the *hatsumon* ("are the rectangles 3cm × 5cm and 5cm × 7cm of the same form?") and of stimulating and formulating students' conjectures and reasoning about that problem, a point also strongly confirmed by the literature (e.g., Stigler and Hiebert, 1999, p. 92f). The problem is also the key focus of the teachers' planning discussions to carefully select and prepare the *hatsumon* according to their estimation of students' needs and capabilities, as well as the needs and potentials of the subject matter at hand. We give more details of how this is realised in the lesson in the next section.

The above five points do *not,* of course, constitute an exact model for every mathematics lesson in Japan. In the present lesson plan, *matome* is not explicitly foreseen, and it is not present in the lesson observed. According to the lesson plan, the lesson should finish with individual work to be used as a starting point in the following lesson; because the *neriage* took up more time than expected, that task is assigned as homework. The lack of *matome* could be explained by the "appetizer" nature of this lesson, which has the purpose of making the students reflect mathematically, rather than to arrive at precise methods or conclusions.

3.3 Students' strategies and reasoning

We now describe in more detail the students' reasoning with reference to the strategies anticipated in the prior discussion and to their meaning in terms of the proportionality. The students' reasoning here is an outcome of their interaction with the milieu in the phase of *kikan-shido* related to the *hatsumon* described in Section 3.2, and led by the teacher in the phase of *takuto* (integrated in this case with *neriage*).

The teacher first calls on students who, he has observed, favour the hypothesis of *same form*. The first, Shinya, explains this using essentially *breadth minus height* as an invariant:

> Well, because the side which is longer, one makes a subtraction of the height, and // one has 2. And, in the one to the right [on the board, as drawn by the teacher] the length minus the height is again 2. Thus, they are of the same form.

A similar explanation is offered by a second student, who says that, as one has added 2cm to both sides, "the difference therefore stays the same". The teacher then leads the students to say that the same reasoning applies to the case of the two squares (where the difference is 0). These explanations may be seen as variations of explanation "b." foreseen in the lesson plan and, to some extent, encouraged by beginning with "extending" the first two squares, then the rectangles.

Some students now claim that the two rectangles do not "appear" to be the same (form), but the teacher puts these claims aside for a while and calls another student, Oto, to the blackboard. She draws the diagonals in the two rectangles along with two angles in each of them (in fact, a triangulation in the sense of Sec. 1.2), and claims that they are the same. Some students insist the angles are different, and after a heated discussion about this, a student says:

> Because the angles are different, therefore they are of different form.

This may be identified with the Euclidean criterion for non-similarity stated at the end of Section 1.2, and with approach "a." as foreseen in the lesson plan. However, Oto maintains that both angles are 35 degrees, while other students almost scream "she made an error of measurement" and that the right hand angle is really 31 degrees.

One of the louder students, Ichiro, insists that they are "absolutely different" and adds: "Well, first we must put the measures at equal size." The teacher finally concedes to let him come to the board and explain his idea. He begins a complicated reasoning, while writing on the board (see Fig. 1):

3 cm and 5 cm, the smallest common multiple is 15 and 15 // 5 times and 3 times // so // this one is 5×5, so 25. And that one is 7×3, so 21. The measures are different. So they are not of the same form.

$$3 \xrightarrow{\times 5} 15 \quad 5 \times 5 = 25$$
$$5 \xrightarrow{\times 3} 15 \quad 7 \times 3 = 21$$

Figure 1. The board writing of Ichiro.

This is a first example of proportional reasoning in the class; it is strikingly similar to the first part of the Euclidean definition in Section 1.2, saying that if (a, b) and (c, d) are proportional, then for any integers m and n, we have $ma = nc \Rightarrow mb = nd$ (applied here with a=3, b=5, c=5, d=7, m=5 and n=3). Since in fact, with these numbers, $ma = nc$ but $mb \neq nd$, the two pairs are not proportional.

Not surprisingly, this is not immediately grasped by everyone in the class, even the teacher (who probably understands) claims not to understand. He asks another student if he understands, and as he says, “yes, I can almost see it”, he asks him to explain it. After a few further explanations, the teacher sums up (writing this time next to rectangles on the board):

Thanks. This time, you see what Ichiro said? He wanted to say that one puts in equal measure the height of this rectangle and the height of this rectangle. So, if we put them to 15cm, they are comparable. If this becomes 15cm, it’s 5 times as big, so the measure of the breadth must also be multiplied by 5cm, so it becomes a rectangle of breadth 25cm. And this one, if we put the height to 15cm, it means it’s 3 times, so the measure of breadth must also be multiplied by 3, so it will be 21cm. So, they are different rectangles. That’s what Ichiro wants to say.

Notice that this explanation was not foreseen in the lesson plan, and it comes out in the subsequent meeting among the teachers that they were really surprised to find it.

In class, the discussion goes back to the question of the angles between the breadth and the diagonal. After discussing their values for a while, the teacher calls on yet another student:

But, Satoshi, he made a table, didn’t you, Satoshi? Could you quickly write it on the board?

Height	3	5
Breadth	5	7
	1.666	1.4

Figure 2. Satoshi’s table.

The student draws up the table shown in Fig. 2, however the teacher asks another student to give the last number (1.4) before allowing Satoshi to write it. After drawing the table, the following dialogue between them:

T: So, Satoshi, what is you conclusion? They are of the same form?
S: Different form.
T: Different form? Why?
S: Because the ratios of the factors are different.

The table is another example of proportional reasoning, using the fractions a/b and c/d to determine whether (a, b) and (c, d) are proportional (approach e. predicted in the lesson plan). This is closer to the idea of “slope” related to the dynamic definition, although clearly no algebra or variables are explicit. We note that this argument remains somewhat isolated and unfinished; to relate it to the intuitive idea of comparing the angles between diagonals and sides would require good knowledge of the tangent function and its inverse, which is clearly out of reach here. After discussing the meaning of "ratio of the factors", which appears odd to some students, the teacher notices that time is up and gives the assignment for homework, as explained before.

In this way, the students came up with multiple strategies in the research lesson, as the teachers expected in the prior discussion. All of them closely relate to the idea students have on the idea of "same form". And these are not either correct or wrong, but different ways to think about "same form". This is a point of the lesson that the teachers expected. What made this happen was the deliberate design of milieu (choice of ratio 3:5 and material milieu) and also the didactical contracts we analyse in the next section.

3.4 Didactical contracts

We focus now on that part of the lesson where students explain their conjectures and reasoning.

The dialogue between Satoshi and the teacher is characteristic of the way in which the teacher refrains from favouring one of the two options in play, to make sure the students' conjectures come from them. He often asks students to repeat or reformulate their explanations, and asks a student to repeat or explain what another student just said. Thus, we may say that after launching the problem, he keeps strictly to serving two purposes: urging the students to explain themselves, to listen carefully to other students, and to explain what others say. This seems to be accepted by the students, who never ask the teacher for "the answer", even at the end of the lesson. On their side, they give careful explanations and discuss animatedly, but in a friendly manner, when they do not agree.

We identify here a firmly established *micro-contract* for the distribution of responsibilities: students and teachers know what they may or must do within the crucial phase of *neriage*. These conventions are probably familiar to the students from other lessons with this teacher. The micro-contract seems crucial to explain the variety and quality of the reasoning put forward in the class. It is not entirely implicit for the teachers. In the lesson plan, it can be traced in the instructions for the *neriage* phase, where we read: "place a great deal of importance on students' explaining of peers' ideas". Also in the preparatory discussion, one teacher says: "Some students will insist they are of the same form. One must make them think about why other students think they are different".

The existence of this micro-contract, of which we have only outlined some striking aspects here, is a condition for the ongoing development of a *meso-contract* which concerns the criteria according to which the problem could be solved, i.e. what knowledge (meaning, criteria, rules) should be developed about the notion of "same form". One event has a strong influence on this: the teacher provides a strong hint, after formulating the problem, as he recalls what a certain student said in the previous lesson (the first in the sequence, cf. Sec. 3.1):

> T: If the measure of the angle is the same, the form is the same. It was Takuya who said that yesterday, right? We'll see if they are of the same measure, if they are of the same measure // You have five minutes, five minutes. OK, think about it. You can do everything you want. Think how to find out.

We notice that this intervention of the teacher is foreseen neither in the lesson plan nor in the preparatory discussion.

This intervention largely determines the meso-contract for some students, who focus on searching for some angles that will settle the problem. Oto's strategy, as presented on the board in the research lesson, would be an effect of this part of the meso-contract. As we have seen, other criteria emerged as well, some of which appear to be more convincing and viable for the sequel. Later on, these elements will be crucial for the establishment of a *macro-contract* related to the wider objectives of the sequence, which seem mostly out of sight for the students in this lesson.

A less desirable possibility for the meso-contract was anticipated and extensively discussed prior to the research lesson : namely that students might insist on the "same form" answer by simply saying that both figures are rectangles. But in the actual lesson, this approach did not appear.

The last student intervention (Satoshi's) can be interpreted as an attempt to comply with a more general objective, beyond the meso-contract related to "same form" figures. Satoshi does not grasp, out of thin air, the idea of the table and the notion of "ratio of factors", but probably associates to readings in the textbook or private studies out of school. This search for a macro-contract could be seen as an attempt to bypass the micro-contract of "open inquiry".

4 A comparison of two didactical designs

We now turn to a comparison of two didactical designs. To do so, we represent the process of lesson design as in Fig. 3. The research lesson of "same form" is, as we have seen, realised in a *lesson study* and based on elements of open approach (although these were implicit). The "Puzzle" situation to be introduced below is realised as part of a *didactical engineering*, based on the idea of fundamental situations in TDS.

Theoretical basis	Design formats	Realised lesson
Open approach	Lesson Study →	"Same form" (Research lesson)
Fundamental situation	Didactical engineering →	"Puzzle" situation

Figure 3. Process of realisation of a lesson in each didactical design

In this section, we first analyse the similarity and difference of the two lessons realised with two different approaches in terms of the milieu and the didactical contract (Sec. 4.1), then compare the two didactical designs at a more general level: theoretical basis and design formats (Sec. 4.2).

4.1 "Puzzle" situation vs. "Same form" situation

The famous *puzzle situation* is part of a sequence of 65 lessons on fractions and decimal numbers, which are described in detail elsewhere (Brousseau and Brousseau, 1988; Brousseau, 1997, Chap. 4). The lessons preceding the puzzle situation concern the multiplication of fractions and decimal numbers, as well as some work with successive magnifications of geometric figures.

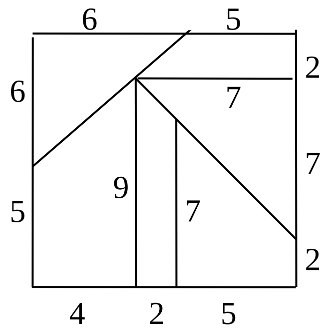

Figure 4. A puzzle (Brousseau, 1997, p. 177). The numbers indicate measures in cm.

In outline, the puzzle situation is based on the following problem: given a “puzzle” like the one shown in Fig. 4, determine how to enlarge it such that the side of length 4 cm becomes 7 cm. The situation is organised as follows: first, students are divided into groups of 4-5, and each group is given a carton copy of the puzzle shown in the figure. The assignment is: each member in the group must enlarge at least one piece, or a pair may enlarge two pieces. Then, the new (larger) pieces must be assembled to a puzzle like the original one.

It appears from experiments made by Brousseau and his colleagues that despite careful preparation in previous lessons, students spontaneously tend to construct the larger pieces by adding 3 cm to all known sides (since 7 cm is 3 cm more than 4 cm). Other methods are then tried out. The crucial point is that *only the multiplication of all sides by 7/4 will work* in the sense that any other method will result in pieces which cannot be assembled. The objective milieu, which consists of the puzzle pieces together with the task on enlargement, offers an adidactical milieu with a visual feedback (gap between pieces) that informs whether or not a strategy is correct, without the teacher’s intervention. That is to say, the refutation of incorrect methods is *built into the objective milieu* and does not depend on the reasoning or convictions of the students. Only one model for magnification will work – the target *proportional model* which is a practical version of the fundamental theorem about similar figures (cf. Sec. 1.2). Other models can be tested, but are refuted by experiments in the milieu. This means that the situation, based on this objective milieu, is a fundamental situation for the proportional model. In the realisation described by Brousseau (1997, p. 177f) the micro-contract (regulating what is at stake at a given moment) changes almost automatically with the students’ discovery of the constraints of the milieu, as they try to fulfil the meso-contract: the enlarged pieces should fit together.

This cannot be said about the situation considered in the previous section. The objective milieu consists in the rectangles A-D as shown in Fig. 5, together with the question of whether C and D are of the same form (after agreeing that A and B are so). Despite the lesson preceding the research lesson, the students do not have a precise definition of the meaning of “same form”, and this results from a deliberate aim of the lesson study team: “to produce as many ideas among the students as possible”, as one of them says in the preparatory discussion. Students are urged, in the beginning, to base their conjecture on their immediate (visual) impression, and then try to find an argument. While the micro-contract is quite stable in this case, as explained above, the students have to search for the meso-contract. The objective milieu does not force or guarantee a certain outcome. The milieu is not adidactical in the sense that the objective milieu does not necessitate a particular strategy, unlike in the puzzle situation. Indeed no definite conclusions are reached in the lesson (although this was the teacher's expectation). Of course, a certain convergence of ideas is to be expected later on, as results from the assigned homework are taken up.

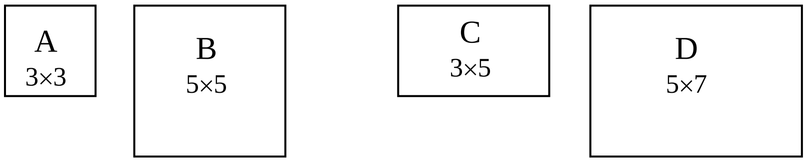

Figure 5. Squares and rectangles from the research lesson.

In both cases – especially in the research lesson – the additive model is expected to come out, with the aim of realising that it is not adequate for describing or constructing figures of the same form.

The situations offer different means for doing so and as a result, orchestrate in different ways the two crucial aspects of mathematical knowledge at stake: criteria for "same form", and the multiplicative model as a way to achieve or check them).

On the one hand, the puzzle situation offers experiential refutation through a conflict between hypothesis and reality; the notion of "same form" remains implicit, built into the material milieu of the puzzle. The multiplicative model there is used as a tool (in the sense of Douady, 1986) to provide a product that stands the test (of leading to the larger puzzle).

On the other hand, the Japanese research lesson offers (lengthy) negotiation about the appropriateness of various definitions and criteria for "same form". The multiplicative model is expected to appear as a tool to justify the answer (same form or not) without any feedback from an adidactical milieu, and at the same time the model itself as an object is formulated and tested in the negotiation.

Part of the difference comes from the larger mathematical complexity of the puzzle as an objective milieu: there, proportionality concerns 3-tuples and 4-tuples (the sides in irregular triangles and quadrilaterals), some of which are unknown to the students. But the pieces may still be constructed from the known sides; and hypotheses are simply tested using those sides as input. Notice that this experimental validation leads implicitly to a view of proportionality which is closer to the *dynamic* definition; indeed, Brousseau (1997, p. 176) analyses the target knowledge in terms of linear functions.

4.2 Fundamental Situations in Didactical Engineering vs. Open Approach in Lesson Study

In Sec. 3, we used TDS to analyse the Japanese research lesson. Viewed in this way, the rationales and learning objectives of the lesson may not be fully transparent, as the adidactical part seems less important than the phase of *neriage*, directed by the teacher. In order to explain the historical basis of the research lesson – and to achieve a comparison at a more general level – we now turn to outline a theoretical framework of Japanese origin.

The "open approach" method for teaching mathematics was developed since the early seventies, in a joint effort of Japanese mathematics educators and teachers (cf. Nohda, 2000). The main idea is to organise a lesson around an “open ended problem”, which are characterised by having multiple correct answers (Becker and Shimada, 1997, p. 1), in order “to foster both the creative activities of students and the mathematical thinking in problem solving simultaneously” (Nohda, 1991, p. 32). This means that the focus in mathematics teaching shifts from training the use of certain methods, to developing a multitude of methods and reasoning supporting them (often referred to as “higher order thinking”). Similar trends exist, of course, in many other countries, but the Japanese context is special both for the richness of examples developed (see e.g. Becker and Shimada, 1997), the efficient means of diffusion (including teacher training, textbooks and national curricula), and the general adherence of teachers and educators to pursuing this method of teaching. In this sense, the research lesson presented above can be said to be implicitly based on the open approach method. While the problem involved seems, at first sight, closed (same form or not?), the real problem is not to give an answer, but to provide convincing arguments in favour of it (this is what we identified as the stable micro-contract in the situation).

When looking globally at the two didactical designs described in this paper, several similarities can be easily identified. Each of them is rooted in significant theoretical ideas (open approach or fundamental situations) and there is a format to implement these theoretical ideas in a real lesson. For the open approach, it is the lesson study (although the latter is not specific to the open approach method, as it existed even before the open approach method was invented, cf. Isoda, 2007). For the fundamental situations (or more broadly TDS) it is didactical engineering, a process of lesson design developed in France (cf. Artigue, 1992). Both of the designs emphasise the social interaction and independent thinking of students. Both formats for design require quite similar kinds of analysis, including anticipating student strategies and revising the design in an experimental cycle.

However, when closely looking at these two formats, the differences between them are perhaps more significant, both as regards their principles for “good” lessons, and as regards their objectives.

In the previous section (Sec. 4.1), we have shown, in two concrete cases, the difference in the nature of milieu and the difference in didactical contracts regulating social interaction within each lesson. These are results of the underlying principle of lesson. In fact, what was expected to be realised in the real lesson with the open approach was to pull out from students multiple ideas and strategies that relate to the target mathematical knowledge. There is no hierarchy among them. The point of the lesson is that students develop and express different approaches to the problem, and reflect on their own ideas by seeking to understand those of their peers. This is a reason why, in the research lesson analysed in Sec. 3, the phase of *neriage* is more important than the adidactical situation where students work independently on the problem. By contrast, in a fundamental situation, the adidactical situation is central, since the aim is for students to construct the one and only “winning strategy” through interaction with the objective milieu. It is in this process that the target mathematical knowledge appears, and this is what the didactical engineering expects and tests in a realised lesson.

This difference of the status of strategies in principles for lessons can be explained by the different purposes of student activity in each approach. In a fundamental situation, they should lead to the personalisation and institutionalisation of a target mathematical knowledge (*savoir*), consistent with the “official” mathematical knowledge. The fundamental situation is a specific epistemological model of a target mathematical knowledge. In the open approach, the aim is for students to apply and test their mathematical knowledge, through two main processes: the process in which some conditions and hypotheses from a “real world” problem are formulated mathematically; and the process of generalisation and systematisation after solving a problem (Becker and Shimada, 1997, pp. 4-7). These processes (associated with open ended problems) are stressed against other processes (e.g., solving closed mathematical problems). In brief, mathematical activities in terms of the open approach may lead to experiencing mathematics in a broader sense than a fundamental situation.

Also at the level objectives, lesson study and didactical engineering differ significantly. The activities in the lesson study are oriented to develop and improve a lesson from the perspective of the people who participate in it. In this process, teachers develop professionally. Unlike lesson study, didactical engineering (based on TDS) aims to establish scientific knowledge: the lesson is realised to confirm the conditions for learning which are anticipated in the *a priori* analysis of the target knowledge and previous experiment. In short, didactical engineering proposes a *systemic* approach to research on the conditions for learning mathematics, while lesson study proposes a *systematic* approach to developing mathematics teaching practice.

5. Conclusion

In this paper we analysed a didactical design for a particular lesson on proportionality in Japan, and compared it with a famous didactical design ("the puzzle") invented in France. At first glance, the two may look similar, despite their separate origins. However, using the detailed analytic tools of TDS, we found several differences in terms of the principles for lessons to be actually realised and the objectives of design in the underlying formats. Based on the differences and similarities identified in this paper, we finally address the question raised of how they may be brought to interact.

First of all, we have seen how the theoretical basis of didactical engineering (TDS) may be used as an *analytic tool* to describe significant differences between the two lessons, regardless of their origin. This confirms that TDS is not bound to bear on the designs which enabled and formed it initially. However, we also noted that some aspects of the Japanese lesson – associated with the method of open approach – require a broader conception of mathematical activity than what we find in TDS, at least in the usual model of didactical situations. This might, in turn, add to the scope of scientific research within the framework of didactical engineering.

On the other hand, for the purpose of teacher led development of teaching practice, lesson study shows a potential of what teachers could take into account or anticipate and how they may develop their practice in shared, systematic ways. The collaboration between researchers and teachers in such a format – which may be transposed to other cultural settings, albeit with significant adaptations – could lead to new and more practice oriented forms of "didactical engineering", if retaining at least parts of the theoretical basis (TDS). Of course, integration of research perspectives into practice is always a delicate issue. However, as our analysis in Sec. 3 illustrates, teachers' principles for lessons (which are often implicit in lesson study) may be articulated and examined more explicitly by means of TDS, as regards what aspects of mathematical knowledge are at stake and how different elements in the lesson design could affect students' learning. This could fertilise teachers' practice in lesson study.

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