THE NATURE AND ROLE OF PROOF WHEN INSTALLING THEOREMS: THE PERSPECTIVE OF GEOMETRY TEACHERS

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We report preliminary results of research on the underlying rationality of geometry teaching, especially as regards to the role of proof in teaching theorems. Building on prior work on the classroom division of labor in situations of "doing proofs," we propose that the division of labor is different in situations when learning a theorem is at stake. In particular, the responsibility for producing a proof stays with the teacher, who may opt to produce the proof in a less stringent form than when students are doing proofs and who may do so for reasons other than conferring truth to the statement. We ground this claim on reactions from experienced geometry teachers to an animated representation of the teaching of theorems about medians in a triangle.

INTRODUCTION AND THEORETICAL FRAMEWORK

Classroom instruction relies on a basic division of labor vis-à-vis the knowledge at stake (the teacher teaches, the students study, and the teacher attests to their learning) that gets further specified (in terms of who does what and how) depending on the particular kind of symbolic goods that are at stake. With the expression "instructional situation" we refer to each of the systems of norms that organize usual transactions between work done and knowledge claimed as taught or learned. By a "norm" we mean a central tendency around which actions in instances of a situation tend to be distributed.[1] Earlier work has studied the situation of "doing proofs" in American geometry classrooms, in terms of its division of labor and its temporal entailments (Herbst & Brach, 2006; Herbst, 2002). We have further hypothesized that as teacher and students participate in an instructional situation, they hold themselves and each other accountable for responding to the presumption that they should abide by those norms. Teachers, in particular, make use of a practical rationality, a system of dispositions, categories of perception and appreciation that allows them to handle the presumption that they should abide by a norm (Herbst & Chazan, 2006). This paper explores the practical rationality that geometry teachers invest when handling the norm that a new theorem needs proof: What are the categories of perception and appreciation that matter on the decision that a proof is needed and on the appropriateness of the proof produced? The importance of this question lies in the fact that in spite of the fact that in mathematics every theorem has a proof, in American geometry classrooms some theorems are proved but others are not.

Our interest is the nature and role of proof in the geometry class. Herbst & Brach (2006) studied the division of labor between teacher and students in the instructional situation of "doing proofs" in which students' engagement in proving is exchanged for
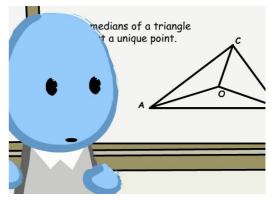
claims on their knowledge of how to do a proof. In those situations teachers are responsible for identifying the task as one of doing a proof, for setting of initial conditions, and for providing the statement to be proved. Students are responsible for orderly producing the statements and reasons that constitute the proof. Each of those "statements" is often a description of a property of the figure represented in a diagram or a transformation of a prior statement, supported by a previously known "reason"—a theorem, postulate, or definition. Yet, the proposition proved—the claim that in the given conditions the conclusion is true—is often of little consequence: Being stated as a true statement about a specific diagram, neither the teacher is responsible to teach it nor students must demonstrate they know and can remember it.

A look at the work of mathematicians suggests that proofs might be present in other occasions of classroom life: when theorems, those general, consequential statements that collect knowledge for further use are installed in the public knowledge. Our inquiry as to the nature and role of proof in geometry instruction took us then to examine another instructional situation, which we call "installing a theorem" and which we define provisionally as the system of norms that regulate what teacher and students need to do to be able to claim that the class knows a new theorem.

METHODOLOGY WITH NOVEL TECHNIQUE

To pursue our interest on the normative aspects of teaching, rather than on the preferences and beliefs of particular individuals, our project gathers data from groups of experienced geometry teachers confronting together representations of teaching that showcase instances of the instructional situation we want to learn more about. We use a novel technique to gather data on teachers' practical rationality. We create stories of classroom interaction and represent those stories as animations of cartoon characters (Fig. 1). These

Fig. 1. A theorem on medians

characters interact in ways that might or might not be common in American geometry classes. They showcase instruction that straddles the boundaries between what we hypothesize to be normative and what we expect practitioners would consider odd. The representations of teaching are shown to the participants of monthly study groups composed of experienced teachers of high school geometry who meet for three hours every month to discuss one or two stories. The discussion among teachers, where they point to odd or intriguing moments in the story, suggest alternative stories, or bring concurrent stories of their own, is therefore a main source of our data. This paper focuses on discussions of the story called "Intersection of Medians" from two of those sessions. We describe the teaching represented in the animation to demonstrate how a story embodies hypotheses about the normative in teaching, and to ground anticipations of participants' reactions which we examine after.

ANALYSIS OF THE ANIMATION "INTERSECTION OF MEDIANS"

The animation "intersection of medians" deals with two theorems about the medians in a triangle. Theorem 1 states that *"the medians of a triangle meet at a point."* Theorem 2 states that "if O is the centroid of triangle ABC then the areas of triangles AOB, AOC, and BOC are equal." The following phases of the instruction represented in the animation describe the different ways in which those theorems are installed.

Phase 1: Defining median and conjecturing Theorem 1
Students are reminded that they know perpendicular bisectors, angle bisectors, and altitudes of a triangle. Medians are defined. The teacher invites a conjecture about medians and students propose that they meet at a point. Theorem 1 is introduced without proof. Centroid is defined.

Phase 2: Conjecturing
The teacher invites students to conjecture a property of the triangles made by the centroid of a triangle (Fig. 1). After some trials a student proposes that they are equal in area and the teacher writes Theorem 2.

Phase 3: Presentation of proof
Teacher presents a proof for Theorem 2 by subtraction of equal areas (see Fig. 2 and Herbst, 2006, p. 324). Students answer focused questions whereas the teacher steers the argument, writing it as a paragraph (omitting reasons and some statements "for the same reason").

Phase 4: Verification activity
The teacher hands a sheet with a triangle for students to find the centroid and measure the areas of the three triangles. They collect measures.

Phase 5: A new conjecture
Students are asked to look at the six triangles made by three medians inside the triangle and invited to make a conjecture. The conjecture that they are equal in area is disputed on account of perceptual differences.

How the theorems are installed in the animation

Conjecturing and corroborating the theorems. Theorem 1 is stated in analogy with other concurrency properties for segments in a triangle and admitted true without proof. In particular, the possibility that three medians might meet in three points is never entertained. This development conforms with customary practice in American geometry classes where all theorems are not usually proved. We anticipated whereas this oddness might prompt a mathematically educated observer to comment, experienced geometry teachers would not react to the installation of Theorem 1.

The installation of Theorem 2 is also odd but in a different way. The teacher invites students to conjecture the theorem providing only a diagram that students can look at (see Fig. 1). Whereas the mathematically educated observer might be able to use the definition of medians to enrich the representation, creating other resources with which eventually produce a reasoned conjecture, the resources provided by the animated teacher are at odds with what geometry teachers customarily afford when they give

students opportunities to make a conjecture about metric properties in a diagram. The animated students are not given the chance to measure dimensions and calculate areas until after the proof has been done. When they produce the conjecture they grope from claim to claim until they succeed, leaving the impression that they are using the teacher's responses to prior guesses as resources to make a conjecture. We anticipated that this would frustrate our participants, and that they would suggest an earlier exploration focused on measuring areas of particular triangles. The empirical verification of the theorem conducted after proving would not only be unhelpful for conjecturing but would also risk casting doubt on the proof. (In the animation the area found for one triangle is slightly less than the other two and the animated teacher brushes that off as a measurement error.)

Characteristics of the proof produced. Fig. 2 contains the finished proof of Theorem 2 as shown on the board. Several characteristics are odd for those who are familiar with how proofs are usually written in geometry classes. Notably, the proof is written in a paragraph rather than in two columns of statements and reasons (Sekiguchi, 1991). The proof does not restate the given or spell out all statements, and it states none of the reasons. Additionally, the proof does not build on congruent triangles, which is the customary way in which proofs are done in geometry classrooms (Herbst, 2006). We want to understand what elements of the teachers' rationality are brought up by our study participants to notice and appraise the decisions made by the animated teacher in proving this theorem. The question was particularly important to ask apropos of this animation since, as the proof did not stage work similar than what students would do when "doing a proof," we anticipated teachers would not necessarily warrant this proof as an example of "how to do a proof."
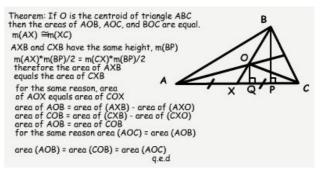

Fig. 2. Proof given in the animation

The division of labor in "intersection of medians." In the animation one can note several actions by teacher or students that straddle the boundaries of what is normative in geometry classrooms when students are engaged in "doing a proof." Not only the teacher writes each statement but also he authors what he writes, which appears often as something that students could not have produced. Unlike in a situation of "doing proofs," key elements to be used in the proof (such as the altitudes BP and OQ; Fig. 2) are not provided at the onset but called up as needed. Likewise, students use the diagram to produce statements that would ordinarily feature in the proofs they are used to do (e.g., OB is congruent to itself), but these are not taken into the argument. Whereas students partake of the production of the proof by uttering responses to the teacher's questions, those utterances serve at best as indication that some students are following the teacher (and others are not), not that they are producing the content of the proof the teacher writes. We thus conjectured that if study group participants were to

positively appraise the animated teacher's decision to prove Theorem 2, they would invest categories of appreciation and perception different than those invested in managing the situation of doing proofs. We expected that those comments might help explain on what grounds some theorems are proved whereas others are not proved in high school geometry classes.

ANALYSIS OF STUDY GROUP SESSIONS

The data gathered consist of videos of study group sessions and their transcripts. By identifying odd aspects of the animation, our prior section identifies particular moments in the story where participants' comments could be expected. We provide illustrative data from two sessions (121405 and 040406) in which the animation was viewed, discussed as a whole, and commented after stops requested by participants.

All theorems need not be proved. Our participants confirmed that, from their perspective, proofs need not always be given, which obviously sets the situation of "installing theorems" apart from that of "doing proofs" in which a proof is required. Participants didn't object to the statement of Theorem 1 without proof and actually manifested that they had never proved Theorem 1 in their classes. Furthermore, some of them expressed that the proof for Theorem 2 might not be needed either. When "installing theorems", proving has less priority than introducing the theorem itself. Some of participants expressed this clearly for Theorem 2:

Tina	Yeah if you were choosing to do the other one, I mean—you have a limited amount of time for every theorem they're gonna be given. We don't have time to prove every single one of them. (121405)
Denise	If you know the theorem you would be able to use it. Sometimes—well all the time—you don't really have to know how to prove everything to be able to use it. So as long as they can use it. (121405)
Glen	It would ultimately be more important to go through that discovery process you just talked through than to go through that proof that none of them are gonna understand. […] (121405)

The reasons given stress that students need to be able to use a theorem (e.g., in calculations), that students may instead benefit by knowing where a theorem comes from (which presumably the proof does not provide), that understanding proofs is hard for students, and that proofs are optional investments of time for the teacher.

Corroborating a theorem. From those comments, we can see that, unlike in the usual practices of research mathematicians, the truth of a theorem is not really established in a geometry class by showing the existence of a proof. This may not surprise those who have characterized traditional mathematics teaching in broad strokes as relying on teacher and textbook authority (Smith, 1996). However, teachers in our study groups did allocate value to raising students' degree of conviction (or epistemic value; Duval, 1991) of a statement. They reacted to the animation on the point that the conjecturing process did not allow students to believe the truth of the theorem:

Megan	Before they start that theorem, they should be believers. Why would the areas be the same? Why does that make sense? Why would I believe that? […] (121405)
Lynne	But just can we even have an example that we think the conjecture is even true? Because the teacher just said 'yup the area one is going to work.' And I think the kid wanted a little verification there, like are the areas really equal? (040406)
Ester	[At] the end he has them measure them. Maybe, say, why don't they measure them first. (040406)

If teachers may not consider that the proof of a theorem is always needed but they value students' thinking to the point that they may spend time on (possibly empirical) work to build conviction of its truth, the question still remains as to what roles a proof may play the proof is provided. One reason may be to enlighten students in regard to the form a proof may have.

Form. The animated teacher called the proof he did a "paragraph proof" and took to do it deliberately, as an alternative to the "two-column proof" that students were used to. Participants' reactions to the form of the produced proof were positive:

Carl	I got the sense from his paragraph proof, which I liked actually […] one of the things I like about proof is that it's really just, um, documenting information that we're trying to keep track of. […] well not by the end of the proof chapter, but certainly by the time we've done a whole bunch of proofs, […] I'm really just more inclined to just say hey, just, you know, get that, get that down, don't forget about that triangle. […] (040406)
Megan	I guess I think I'm agreeing with him that now at this point of the year, I guess I'm relaxing on that and saying I want you to just write out a logical argument. Because some of them, I think the proof format ends up —they get so bogged down in the format. […] (040406)

Hence, a proof without all the detail might be acceptable as the year progresses. Students need to experience getting the important (as opposed to all) information needed for the argument, relaxing the strictures of how to write it and yet keeping it a logical argument. A proof of a theorem could thus be an opportunity for students to learn the difference between a logical and a detailed argument, but in that case the shares of work producing the proof might be different than when "doing a proof."

Proof production. In the animation, the teacher presented the proof. Students intervened little, and rarely introduced elements for the argument. Participants were ambivalent on this matter. Some positive appraisals included:

Karen	[…] there's this sort of thing where math as a spectator sport is absolutely wonderful. […] And it's like going to art appreciation. (040406)
Lynne	It's like coaching volleyball, you have to take them to a varsity game or a college game so they see the flow of how the whole thing works and then they really want to get back and try those smaller basic things (040406)

In some negative appraisals participants noted that the teacher didn't take into account students' comments while doing the proof. For example:

Denise   […] Because those kids were not doing the proof with him. I mean they were, but they wasn't. […] people were just calling out the answers and he [the teacher] was just waiting on the right answer. (121405)

Marvin   […] you might say to the students, […] how are we going to find areas in this figure, or what triangles do we know the area of? […] do you students see any triangles in here that we could show […] are equal area. Or, what do you […] know about the figure in general? (040406)

Participants' comments note that the class didn't do what is expected of them when they "do a proof." Yet students' "appreciation" or understanding of the proof could also be a legitimate goal for the teacher. However, on that point the animated teacher is found wanting: a teacher should access and support students' understanding better.

A proof connects ideas. A participant also identified another role of proof in the proof given for Theorem 2. Whereas Megan downplayed the usefulness of the theorem itself she valued the proof because it connects some mathematical ideas:

Megan   That particular theorem [is] not very useful on it's own after that. But the proof is actually very useful. I think that's a perfect example of a theorem where the proof's a lot more useful than the theorem is, in the end. Because you're talking to kids about the area of a triangle, and what is the crucial thing to know? I need to know the base and the height, and the fact that the—you have different triangles that have the same base and height and they all have the same area (121405)

Thus teachers' rationality allows for proof as a way to explain and connect ideas, agreeing with what scholars have proposed as desirable (e.g., Hanna, 1990).

CONCLUSION

From the teachers' observations as to the need for corroboration of a theorem, we infer a new element that distinguishes situations of "installing theorems" from situations of "doing proofs." In the latter the truth of the conclusion is not at stake: the problem often asserts that a proof can be done and the work is to show that the givens imply the conclusion. Herbst (2004) has argued that usual proof exercises confront students with a diagram that not only represents the objects in the statement but also is sufficiently accurate for students to be able to produce statements that also describe apparent diagrammatic truths. Theorem 2 illustrates that all geometric propositions cannot be warranted on such descriptive interactions with a diagram. Teachers' reactions suggest that it is important for teachers to achieve, in a different way than by visual apperception, students' conviction of the truth of the statement.

We have located differences in the division of labor between "installing theorems" and "doing proofs": in the former, statements can be unimaginable by students proof, details may be excluded, and a theorem may be established without proof. We may also see, from these results, some inconsistency which might happens in teaching practice across the situations of "installing theorems" and "doing proofs." When "doing proofs" the conclusion cannot be used until proved. However, when "installing theorems," the teacher makes it usable for students who need conviction but may be

better convinced by arguments other than proof. Thus, sometimes a theorem is introduced without proof, and other times its truth is corroborated empirically. Our data suggests that such practices, albeit problematic in that they foster misconceptions well documented in the learning literature (e.g., Fishbein & Kedem, 1982) may respond to teachers' perception of what will convince their students at the moment when they need to appropriate the theorem.

Notes and acknowledgements

[1] The word "norm" is used with similar meaning as Tsai (2004) but with a meaning different than that of Yackel & Cobb (1996).

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