GEOMETRY TEACHERS' PERSPECTIVES ON CONVINCING AND PROVING WHEN INSTALLING A THEOREM IN CLASS

Takeshi Miyakawa
University of Michigan
miyakawa@umich.edu

Patricio Herbst
University of Michigan
pgherbst@umich.edu

This paper advances understanding of instructional phenomena by focusing on instructional situations. We argue that the decisions in managing a situation require a teacher to respond to norms by negotiating dispositions that might contradict each other. We illustrate this by examining the installing of theorems in high school geometry.

Recent research on mathematics learning has called attention to the nature of the situation that serves as context for that learning (Brousseau, 1997; Lave & Wenger, 1991; Schoenfeld, 1998). In our research, we conceive of classroom life as organized by recurring *instructional situations*: frames that allow teacher and students to exchange the work they do for claims on the stakes of teaching and learning. Decisions and actions made by teacher or students not only result from individual thinking or belief, but also respond to the norms of the instructional situation in which those decisions and actions are made. Depending on the instructional situation, different actions may be normative. For example, in the situation we've called 'doing proofs' in high school geometry classrooms, it is normative for the teacher to state the proposition to be proved in terms of a specific diagram (Herbst & Brach, 2006), but in the situation of 'installing' a new theorem, it is normative for the teacher to state a theorem in terms of abstract concepts (Herbst & Nachlieli, 2007).

In this paper, we advance understanding of instructional phenomena by focusing on the norms of those instructional situations as they relate to two different activities which have traditionally been studied in their cognitive and epistemological dimensions: convincing (or bringing someone to a state of belief) on the truth of a statement (Harel and Sowder, 1998) and proving (or establishing the truth of a statement for a given community of knowers; Balacheff, 1987). We argue that a teacher's management of those activities requires them to respond to norms of the instructional situations where those activities occur. A teacher's response to those norms is constructed by decisions and actions that articulate various dispositions that might contradict each other. We illustrate this point examining further the situation of installing theorems in the high school geometry class and drawing from teachers' responses to an animated representation of the teaching of a theorem about medians.

Teaching in Classroom: Practical Rationality

Classroom instruction relies on a tacit contractual relationship vis-à-vis the knowledge at stake: the teacher teaches that knowledge to students, the students study that knowledge with the help of the teacher, and the teacher attests to the students' learning of that knowledge (Brousseau & Otte, 1991). That contractual relationship is more specific (in terms of who can or has to do what, when, how, and to get what) depending on the particular kind of symbolic goods that are at stake. The solving of an equation in algebra, and the doing of a proof in geometry, for example, are both activities that require students to lay down a reasoned sequence of statements. But they differ in terms of the role of *reasons* in those sequences, and in the extent to which those reasons need to be explicitly laid down alongside the statements. What is at stake in both situations is not just a claim on the final statement in the sequence (the statement of what *x* equals, or the statement of the conclusion to be proved) but also a

claim on knowing the "method" of solving an equation in one case, and a claim on knowing "how to do a proof" on the other (Chazan & Lueke, in press; Herbst & Brach, 2006). With the expression "instructional situation" we name each of the various frames that enable teacher and students to bill stretches of classroom work on account of the objects of knowledge they have contracted about. We model situations as systems of norms that organize those transactions. By "norm" we mean a central tendency around which actions in instances of a situation tend to be distributed. We posit that those (mostly unspoken) norms shape a teacher's and her students' actions: As they participate in an instructional situation, they hold themselves and each other accountable for responding to the presumption that they should abide by those norms.

We are interested in the situation of "installing theorems", namely the system of norms that regulates the work teacher and students need to do in order to be able to take for granted that the class knows a specific theorem. We explore two norms that we hypothesize to be characteristic of the situation "installing theorem": (1) students should come to believe the statement asserted by the theorem is true, and (2) for a statement to be a theorem it has to be provable. We investigate how teachers manage their way about those two norms as they act and make instructional decisions about the teaching of a theorem.

As the discipline of mathematics accrues its knowledge, the capacity to show that a proof exists is the sole grounds on which the mathematical community comes to officially believe the truth of a statement (Lakatos, 1976). But a mathematician's belief on the plausibility of a result often hinges on other means (e.g., Pólya, 1954). Mathematical proof is only one of the strategies that might obtain ascertainment and conviction on the part of individuals (Harel & Sowder, 1998). Our focus is neither on the discipline nor on the individual, but rather on the public work done in the classroom. In the classroom, teachers are accountable to mathematics as they propose that a statement is a theorem and also accountable to students as they expect students to take such statement as true. It is thus reasonable to hypothesize the two norms listed above as having a hold on the way the teacher goes about her work teaching a theorem. But to say these norms regulate that work means not that they determine or dictate what a given teacher does; rather a given teacher constructs original actions in response to, against the backdrop of, those norms. The value of modeling instructional situations as systems of norms is that it allows us to study the resources with which teachers construct those actions.

We propose that to construct original actions in response to those norms, teachers, in particular, make use of a *practical rationality*: a system of *dispositions*, categories of perception and appreciation that allow them to notice and value possible actions (Bourdieu, 1998; Herbst & Chazan, 2006). This paper explores the practical rationality that geometry teachers invest when handling the two norms: that students need to come to believe the statement of a theorem, and that for a statement to be a theorem it needs to be provable.

"Convincing" and Proving

In a previous analysis (Miyakawa & Herbst, 2007), we identified the term "believe" as one used by teachers to describe a desirable state of affairs for students vis-à-vis a true statement. We use the term "convincing" to describe the work a teacher might do to make students "believe" a statement is true. Clearly, this "convincing" could conceivably be done in several ways, ranging from mere appeal to authority ("trust me") on one extreme, to the organization of an adidactical situation of validation (Brousseau, 1997) on the other. One conceivable way of convincing a class could be by engaging the class in proving the statement. Is the engagement of students in proving a statement a viable way for a geometry teacher to convince students of the truth of a statement? We seek an answer to that question

based on the rationality that teachers invest when they do their work.

Before proceeding, we clarify the difference between convincing and proving. In both processes, the statement dealt with is the same. However, the end products each process seeks for that statement, proved and believed, are different. Duval (1991) uses the expressions "logical value" and "epistemic value" to describe the different values attributed to a statement. The former expresses the mathematical value of true, false, or unknown. Independent of this, Duval proposes "epistemic value" to represent "the degree of certitude or conviction attached to a statement" by an individual, often expressed with words such as "probably," "impossibly," "certainly," etc. *Convincing* is a process of attributing or modifying an epistemic value to a statement; *proving* is a process of attributing a logical value. Of course, it is conceivable that proving might also help to convince.

The notion of "cognitive unity" has been proposed to gauge the relationship between conjecturing and proving theorems in classroom activity (Garuti et al., 1998). The notion that there exists "cognitive unity" characterizes the case in which there is continuity between the two processes of conjecturing and proving. This continuity is visible, for example, in the use of the same arguments during conjecturing process and proving process (Garuti et al. 1996, p. 113). We use "cognitive unity" to examine the relationship between convincing and proving in terms of the geometric objects students might be asked to work with and how they might be asked to work with them. The data that we use to examine our question on practical rationality consists of teachers' reactions to a teaching episode where a lack of continuity in cognitive unity is described.

Method

To pursue our interest in the practical rationality of teachers of geometry, seen as a collective, our study gathers data from groups of experienced teachers of high school geometry who confront together representations of teaching that showcase instances of installing a theorem (see Herbst & Chazan, 2006). We use a novel technique to gather data on teachers' practical rationality. We create stories of classroom interaction and represent those stories as animations of cartoon characters. These characters interact in ways that might or might not be common in American geometry classes. They showcase instruction that straddles the boundaries between what we hypothesize to be normal and what we expect practitioners might consider odd. The representations of teaching are shown at monthly meetings of experienced teachers. In the discussions that ensue participants point to odd or intriguing moments in a story, suggest alternative, possible stories, or bring concurrent stories of their own collection. We focus on discussions of a story called "Intersection of Medians," and present a result of the analysis on the transcription of the teachers' discussion.

Analysis of "Intersection of Medians"

"Intersection of medians" deals with a theorem about the centroid of a triangle: the intersection of the medians of a triangle, when joined to the vertices of the triangle, forms three triangles of equal area. The installation of that theorem in the animation is hypothesized to be odd for the following reasons. The teacher draws a diagram on the board showing the intersection of the medians and the three triangles and asks students, "What do you conjecture the theorem will say about those triangles?" Students are not given an opportunity to measure dimensions and calculate areas. Merely looking at the diagram from their seats, they grope from claim to claim until they succeed, creating the impression that they are using the teacher's verbal responses to prior guesses, "Close but not quite… Anybody else?" as resources to improve a conjecture. Once the students hit at the right conjecture, the teacher

affirms it ("They have the same area, that's right") and proceeds to produce a proof by himself, requiring only limited participation from students. Empirical verification of the theorem is prompted *after* proving by the teacher.

We expected that experienced practitioners would react to the sequence of events in that story. Particularly, we expected that the temporal displacement of the empirical activity to after the proof would be denounced as depriving students of a resource for conjecturing. We also expected that the location in time of this empirical activity would be denounced as potentially bringing to question the capacity of the proof to build conviction.

We suppose that the measurement of area in each of the target triangles (ΔAOB, ΔAOC, and ΔBOC) would normally be done to convince students that those areas are equal in spite of looking different. Typically a student would measure the area by choosing bases and drawing corresponding altitudes for each triangle, measuring those, and then calculating each area separately using the area formula. There are two options for the choice of bases: either the three sides AC, AB, and BC, or the three internal segments AO, BO, or CO. The choice of a base automatically determines the altitudes. In contrast, the proof given in the animation (Fig. 1) requires students to consider not only the three target triangles, but also the mid-size triangles (e.g., ΔAXB), and the small triangles (e.g., ΔAXO). And to prove that the areas of two target triangles are equal in this proof, it was not necessary to consider the bases and heights of these triangles at all — just the bases and heights of the mid-size and small triangles. The comparison of areas of the target triangles is done by subtracting the areas of the small triangles from the mid-size triangles, not considering any multiplicative relationship between heights and bases in the target triangles. Thus this story represents a continuity gap between the convincing and proving processes in terms of the objects used (triangles, altitudes, and bases). Whereas the conception of equal area at play in the proof is one that deals with area of figures as quantities, the one used in the convincing activity deals with area of figures as a number produced out of multiplying measures of quantities (see Herbst, 2005).[1]
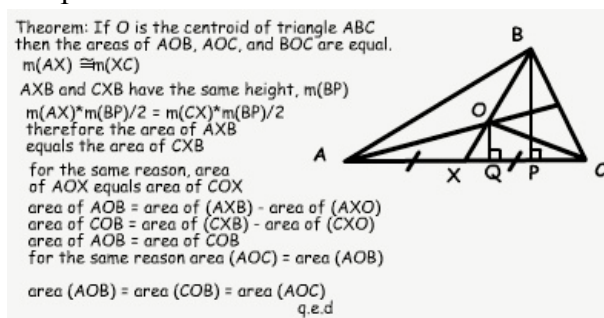

Fig. 1. Proof given in the animation

Results and Discussion

Miyakawa & Herbst (2007) identified the disposition "*it is desirable that students believe the truth of a statement before proving it.*" Participants of study group reacted to the animation by arguing that the conjecturing process did not allow students to "come to believe" the truth of the theorem, and proposed as an alternative that the measurement of areas take place *before* proving. This indicates that teachers value students' thinking to the point that they may spend time on (possibly empirical) work to build conviction of the truth of a claim. We now identify dispositions on the relationship between convincing and proving.

Negative Attitude toward a Gap between Convincing and Proving

Participants identified a gap between measuring and proving. They assumed students would measure the areas of target triangles using the sides of ΔABC as bases.

Ester   I'm not sure it [measurement] would give them ideas about proving. I think it

might confirm to them that it's, whether it's true or not but I'm not sure, I'm not sure if it would help them get any ideas about how to prove.

Megan   Yeah, one drawback though I see, I didn't think of this before, but if he had them do it [measurement] at the beginning, then they'd be looking at those alti—those different altitudes. [...] So then you've sort of steered them in the wrong way for the proof. [...] When you're doing the proof, you're not really looking at those triangles.

For Megan, the measurement activity constitutes a "drawback," because the objects used in measuring — in particular the altitudes and bases of the three target triangles — are different from the objects used in the proof. This is the gap we mentioned above in the analysis of the animation. She also expresses misgivings ("steer them in the wrong way") that might make her reconsider the measurement activity or the proof itself. The disposition we may identify here is "*it is not desirable for a the teacher to steer students in a wrong way.*" This seems to recommend a different course of action than the disposition about conjecturing. On the one hand, since it is desirable that students believe a theorem true before proving it, a measuring activity seems to be advisable. On the other hand, since a teacher should not mislead students, a measuring activity that does not involve the ideas to be used in the proof (or, worse, one that suggests using different ideas) might not be best. These two dispositions suggest a tension in the teaching of this theorem: should the proof be a different one to reinforce what the measurement achieved, or should the measuring activity be left for later to avoid steering students in a direction opposite to that of the proof? And what if the measurement produced areas that were slightly unequal (as happens in the movie)?

Alternatively, a teacher might choose another proof, one that had more cognitive unity with the measuring activity (see endnote 1). This might allow the teacher to avoid the tension between those dispositions. But it might require students to invest different knowledge (e.g., the capacity to look at the same triangle from different points of view, or knowledge of the fact that the centroid divides a median in a 1:2 ratio). We hypothesize that another disposition, such as "*it is desirable that the conceptual complexity of a proof be kept under control,*" is active at the moment of deciding which proof to use, and that might militate against those alternatives. The outcome of such tension might be the decision not to undertake the proof of this theorem at all. Indeed, while teachers acknowledge that every theorem should be provable and could be proved, they also report as a matter of course that not every theorem is actually proved (Miyakawa & Herbst, 2007). The presence of competing dispositions touching on matters of conviction of the truth of statement, cognitive unity, and conceptual complexity, might help explain why some theorems are not proved.

Positive Attitude toward a Gap between Convincing and Proving

We also identified in the same study group session a positive attitude toward the gap between measuring and proving. In particular, Karen liked the proof given in the animation:

Karen   That's what's really cool about the proof. Is that you're proving something, you're proving something about these areas without ever finding the base and the height of these other triangles. [...] Wait how'd you do that? It's magic.

Later in the same session, she mentioned again the benefit of this particular proof.

Karen   [...] How does someone come up with this incredible idea to measure, to figure out that these are the same by doing some sort of subtraction deal? And in that way, you are able to highlight what a beautiful proof it is. You know, I like proofs that go around to the back door.

First of all, for Karen, the proof (Fig. 1) is "really cool" and "beautiful." One criterion of beauty visible in Karen's comment is that the proof can be accomplished without using the objects necessary for the measurement ("without ever finding …"). That is to say, she appreciates the discontinuity between measuring and proving ("magic"), contrary to the disposition noted previously which assigns negative value to that gap because it could cause students' difficulty. From Karen's comment we propose the disposition that "*it is desirable for a teacher to show a beautiful proof.*" This disposition supports the use of measurement in the process of convincing, since it helps bring home the point that the proof predicted such result. But Karen did recognize a possible difficulty students might encounter. The following excerpt of transcripts shows that in spite of the difficulty, the gap is worth considering.

> Karen   It's a roundabout proof. So like that you can't, like if you're stuck on a proof, and you can't figure out how to prove the next step, you back off and go around to other ways to find it. Whereas what happens with the kids is they get stuck and they just, you know, they'll quit or they're, ye--, or they'll yell. But they don't do a whole lot of thinking about how else could I look at this? And we have to keep getting kids to look at how else could I see this? […]

Karen anticipates that students might encounter difficulty and might simply give up on a proof. Despite of this possibility, she allocates value to the process of exploring an argument when students encounter a difficulty in proving: "we have to keep getting kids …" We may identify here another disposition that could be a reason for the former disposition about "beautiful" proof, why it's worth teaching: "*it is desirable for students to bridge the gap between conviction and proof.*" This supports the use of measurement and the proof, and takes into account not only students' difficulty but also the process of overcoming it. It seems that this disposition relates more generally to problem-solving or strategic skills that students need to draw upon when they encounter a difficulty. The need to bridge the gap between what convinces students of the truth of a statement and the proof that such statement is true might be a place in which to develop those skills.

Summary

Different dispositions mentioned in this paper can be summarized as follows (see also Miyakawa & Herbst, 2007). They are activated in response to a representation of teaching in which a gap in the cognitive unity between measuring and proving is identified.

1. *It is desirable that students build conviction of the truth of statement before proving*
2. *It is not desirable for a teacher to steer students in a wrong way*
3. *It is desirable for a teacher to show a beautiful proof*
4. *It is desirable for students to bridge the gap between conviction and proof*

These dispositions sometimes conflict with each other and push teachers to value different actions in teaching. The disposition (2) could push a teacher to avoid proving a particular theorem, so as not to mislead students. On the other hand, (3) and (4) could push a teacher to engage students in appreciating a "beautiful" proof and valuing the work involved in bridging a gap. The proof is indispensable in this case. We may also find that (2) focuses more on students' understanding of the given proof, whereas (3) and (4) focus more on the teaching of specific aspects of proving. We could understand therefore that the matters of conviction of a truth of statement, difficulty caused by the gap of cognitive unity, "beauty" of proof, and problem-solving skill can be taken into account when installing theorems. The participants in our study group allocated various values to these aspects of teaching.

All of these dispositions are not necessarily specific to the situation of installing a

theorem even though they bear on the desirability of giving a proof and on the desirability of a particular proof. (1) is apparently specific to the instructional situation of installing theorems. (2) is a more general clause of the didactical contract that makes the teacher responsible to teach true knowledge to learners; it is only activated in this situation by the concomitant identification of a gap between measuring and proving. (3) might help teachers make decisions related to proving not only when installing a theorem but also when choosing problems where students are expected to do the proof. All of them help a teacher navigate the installation of a theorem.

Notes and Acknowledgements

[1] There are some alternatives for proving this theorem, which might have more continuity in terms of the triangles used between measuring and proving. For example, if the segments AO, BO, and CO are seen as bases of target triangles, any two of three triangles always have the same height and base (e.g., the segment AO as a base of ΔAOB and ΔAOC).

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