OPERATING ON AND UNDERSTANDING OF PROBLEM SITUATIONS IN PROVING

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Proving in secondary mathematics often appears as a solving process of proof problems. This paper presented a model of proving processes which is based on one conception of mathematical problem solving. The aim of this paper is to investigate students' proving processes to check the validity of this model and get insights into students proving processes from the viewpoint of this model. For this aim, two proving processes in each of which a pair of 9th grade students tried to solve a plane geometry problem using a dynamic geometry software will be analyzed qualitatively. The analysis will illustrate the interactions between the students' understanding of the problem situation and their explorations and the importance of deepening solvers' understanding in proving process. It will also show that operating on the problem situation and getting in touch with it is useful throughout proving processes and that solvers' interpretations are influenced by their understanding of the problem situations. One of the merits of the use of DGS will be discussed in relation to such points.
This paper is research oriented and focuses on describing and interpreting students' behaviors in RPP tasks.

1. Introduction

In the secondary school mathematics, many of the proofs may be developed by students as solutions to proof problems. This implies that proving processes can be analyzed as problem solving processes. Nunokawa (2006a) presented a model of proving processes (Figure 1) which is based on his conception of mathematical problem solving (Nunokawa, 2005). The aim of this paper is to investigate students' proving processes to check the validity of this model and get insights into students proving processes from the viewpoint of this model. For this aim, the proving processes in which students tried to solve a plane geometry problem using a dynamic geometry software (DGS) will be analyzed qualitatively. Since their use of DGS might activate their exploration of the problem situation, those date seem appropriate for this aim.

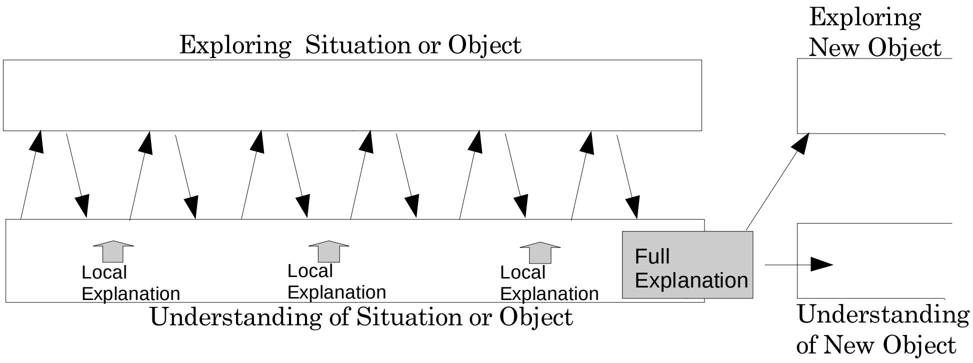

Figure 1 One Conception of Proving (Nunokawa, 2006a)

2. Method

Two pairs of the 9th graders in Japan were asked to solve the following Problem using Cabri Geometry. Each pair used one computer.

Problem : Take an arbitrary triangle ABC. Construct equilateral triangles ΔBAD and ΔACE on the sides BA and AC respectively on the opposite side of ΔABC. Construct an equilateral triangle ΔBCF on the side BC on the same side of ΔABC (Figure 2). Find what a kind of quadrilateral ADFE is.

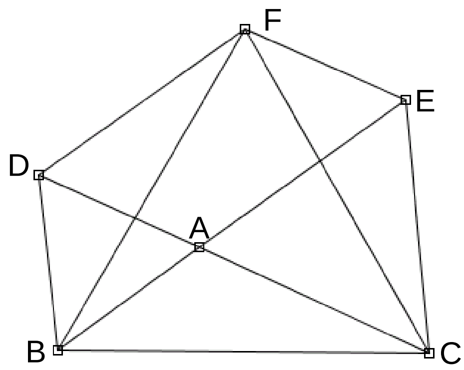

Figure 2 Problem Situation

The students' activities were recorded by a video camera. The protocols including the students' manipulations of the software were made by referring to these video records.

These protocols were qualitatively analyzed to get insight into the proving process focusing on what they found and what they paid attention to at each step of their proving processes and how they came to find the idea central for their proof. Here the figure satisfying the given condition can be considered the problem situation the students needed to explore.

3. Descriptions and Analyses of the Proving of the First Pair

3.1 Proving Process of Pair A (Taka & Nishi)

The students found visually that ADFE becomes a parallelogram and checked it by the measurements of its four sides and its four angles. When the teacher asked why it becomes a parallelogram, they answered referring to those measurements. Being encouraged to use DGS by the teacher, they started to use its functions to explore the situation. When Taka dragged some points, ΔABC became a triangle which looked like an isosceles one. After breaking down this special case, Nishi noticed that ΔDBF and ΔEFC are congruent and checked this by dragging and measurement (the lengths of CE and BD). However, he could not find why these two triangles are congruent at this point. When the teacher asked to share what they had noticed, Nishi only mentioned that the lengths of DA and BA are equal because they are the sides of the same equilateral triangle. Then, Nishi measured the lengths of BC, CF, and BF and looked at the screen. Taka mentioned here that EF corresponded to BD and then AD was of the same length.

When the teacher asked again to share what they had noticed, Nishi stated that three equilateral triangles are similar to each other. After that, Taka dragged the point C so that A was on the side BC (see Figure 3). When Taka dragged C again and this special case broke down, Nishi said, "Wait, wait, that one" and returned the situation to the special case. He moved C downward to break down the special case again. After this dragging, Nishi measured the lengths of AB and AC. Nishi mentioned that ΔABC and ΔDBF are congruent and explained to Taka that they needed to show AC=DF or ∠ABC=∠DBF, as well as AB=DB and BC=BF, to conclude that congruence. Nishi stated that AB=DB and BC=BF because ΔABD and ΔFBC are equilateral triangles, though he also referred to the measurements of those sides.

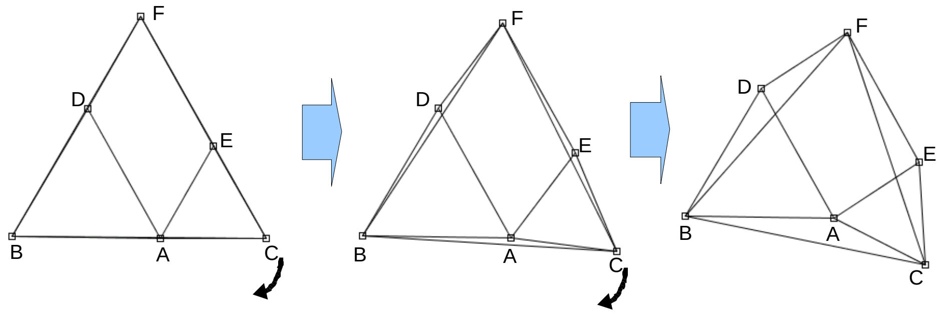

Figure 3 Moving Point and Emerging Triangles

Taka proposed to show ∠ABC=∠DBF referring to AB=DB and BC=BF (and without other reasons). They measured the angle measures of ∠ABC and ∠DBF. When they looked at the screen, Taka stated that BC=BF and AB=DB because ΔABD and ΔFBC are equilateral triangles. They also mentioned ∠ABD=∠FBC=60°. After that, Nishi said that he might find the reason of ∠ABC=∠DBF. He explained to Taka that ∠DBF=∠ABD−∠ABF=60° −∠ABF and ∠ABC=∠FBC−∠ABF=60°−∠ABF and that the same calculation could be applied to ΔABC and ΔEFC. This understanding of the problem situation could lead them to the proof of the conclusion that ADFE is a parallelogram.

3.2 Analysis of the Proving by Pair A

The proving process by Pair A can be summarized as follows: (a) find that ADFE is a parallelogram; (b) dragging test accompanied by the measurements; (c) be aware of AD=EF and AE=DF; (d) breaking down a special case, notice that ΔDBF and ΔEFC are congruent; (e) test it by measuring CE and BD; (f) be aware DA=BA because of the equilateral ΔABD; (g) measure BC, CF, and BF; (h) notice that EF=DB leads to EF=AD; (i) attend to three equilateral triangles; (j) seeing the breaking down of a special case, notice that ΔABC and ΔDBF are congruent; (k) measure AB and AC; (l) be aware of AB=DB and BC=BF based on equilateral triangles ΔABD and ΔFBC, and attend to ∠ABC and ∠DBF; (m) measure ∠ABC and ∠DBF; (n) be aware of ∠ABD=∠FBC=60° and prove ∠ABC=∠DBF; (o) prove their conjecture.

Pair A could prove their conjecture at the end of their solving activity. Their proof was based on their understanding that three triangles, ΔABC, ΔDBF and ΔEFC, are congruent. This understanding was developed gradually during their proving process. They first noticed that ΔDBF and ΔEFC are congruent. At this point, this relationship between these triangles was a mere sub-conjecture. Then they seemed to relate these triangles to the given three equilaterals. Through such an attempt, they understood that EF=BD could lead to EF=AD. This relationship was used in the final proof, even though they did not pay enough attention to ΔABC at first. They also attended to AB and measured BC during this attempt, in relationship to the equilateral triangles. Seeing the breaking down of the special case after this attempt, they found that ΔABC and ΔDBF are congruent. It may be possible that their attention to AB and BC helped solvers pay attention to ΔABC in the figure presented on the screen. Based on this sub-conjecture (congruence of ΔABC and ΔDBF), the students conjectured that ∠ABC and ∠DBF are of the same size. Their attention to these angles, as well as to ∠ABD=∠FBC=60°, might lead to measuring those angles and facilitate their realization of ∠ABF, which connect, for example, ∠ABC to ∠FBC.

This analysis shows that on the way to the final proof, the students gradually found new elements in the situation (e.g. ΔDBF and ∠FBC) and new relationships, and realized the role a certain element could play in this situation. Although some relationships were not fully proved when they were noticed, they seemed to direct next explorations and lead to more deepened understandings through those explorations. These features of this proving process seem to support the model shown in Figure 1.

When the students noticed the critical relationship, the congruence of ΔABC, ΔDBF and ΔEFC, DGS seemed to play an important role. When DGS showed the special case where ΔABC looked like an isosceles triangle and they broke down this special case slightly, Nishi noticed that ΔDBF and ΔEFC are congruent. Nishi realized that ΔABC and ΔDBF are congruent, when DGS showed another special case, where ΔABC was flat, and Taka broke it down. We should note here that the appearance of the former special case and the break of the latter special case seemed to be beyond the students' intentions. These patterns might emerge by chance. It may be one of the effects of DGS that it can produce some cases or highlight some elements which solvers do not attempt to explore in advance (see Nunokawa, 2006b).

We should also note that taking advantage of such emergent patterns seemed to be supported by the students' understanding of the problem situation. Before noticing that ΔDBF and ΔEFC are congruent, they measured and paid attention to EF and DF which are both the sides of ΔDBF and ΔEFC and the sides of a parallelogram ADFE. Before noticing that ΔABC and ΔDBF are congruent, they attended to AB as a side of ΔABD and measured BC as a side of ΔFBC. The students might see the emergent patterns on the screen with such understanding of the situation. It might direct their attention to a certain part of the figures on the screen or provide them with the information to interpret the figures.

4. Descriptions and Analyses of the Proving of the Second Pair

4.1 Proving Process of Pair B (Nogawa & Yama)

They measured four sides and four angles of ADFE and noticed that ADFE is a parallelogram based on such empirical data. They also used a DGS function to check that the opposite sides are parallel to each other. Dragging the point A, they found that D and E also moved because they are vertices of the equilateral triangles. They raised a question why the lengths of opposite sides of ADFE were equal even when the equilateral triangles were of different sizes. At this stage, they attended to AB=AD and AE=AC. Being encouraged by the teacher to use DGS functions further, they replayed the construction. At the step where the line DF was drawn, the students raised the question why DF could be parallel to and of the same length as AE even though two vertices of equilateral triangles were simply connected (see Nunokawa

& Fukuzawa, 2002).

The teacher advised to measure the lengths of the sides, and the students measured the lengths of the sides other than the sides of ADFE whose lengths had been measured before. During this measurement, Nogawa mentioned DB=AB. When they dragged the vertices of ΔABC and the teachers asked what they noticed, they merely mentioned that equilateral triangles remained equilateral. When the teacher asked to make ΔABC a special one which they could name, the students made it an isosceles triangle. Nogawa noticed that the lengths of all the sides except the sides of ΔFBC became the same (8.59cm, Figure 4). Yama stated

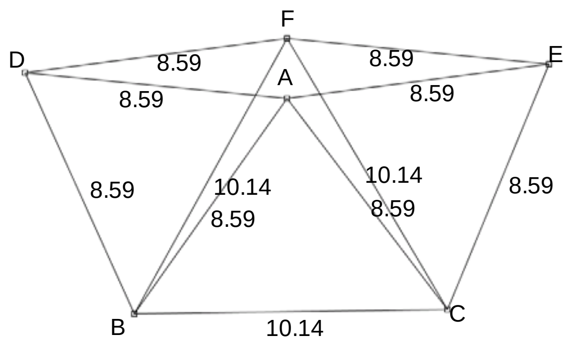

Figure 4 Measurements in the Isosceles Triangle Case

that while he could understand why the lengths of AD and AE became 8.59cm, he did not find why the lengths of DF and EF also became 8.59. They also noticed that ΔEFC, ΔDBF, and ΔABC were isosceles triangles. They interpreted DF and EF as the sides of isosceles triangles and AD and AE as the sides of equilateral triangles. After dragging the point A, they found that ΔEFC and ΔDBF became isosceles triangles only when ΔABC was isosceles. Yama returned the situation to the special case where ΔABC was an isosceles triangle, the lengths of whose sides were 6.87, 6.87, and 10.22. Yama explained why the opposite sides were of the same lengths when ΔABC was an isosceles triangle as follows: (a) the lengths of AD and AE were 6.87 because they are the sides of the equilateral triangles; (b) the lengths of DF and EF were 6.87 because they are the sides of the isosceles triangles ΔDBF and ΔEFC. During his explanation, the point A happened to be moved slightly. Seeing the screen, Nogawa noticed that ΔEFC, ΔDBF, and ΔABC were congruent and Yama found that rotating ΔABC around the points B and C would make it ΔDBF and ΔEFC respectively. They dragged the point A to check their conjecture that ΔEFC, ΔDBF, and ΔABC were congruent. Using this fact, they could show why EF is of the same length as AD, while they referred to the result of “parallel?” command to conclude that ADFE is a parallelogram. Consequently, they could not prove why ΔDBF and ΔEFC are congruent to ΔABC and could not complete the proof of their conjecture.

4.2 Analysis of the Proving by Pair B

The proving process by Pair B can be summarized as follows: (a) measure four sides and four angles of ADFE; (b) find that ADFE is a parallelogram; (c) check whether opposite sides are parallel; (d) drag A and find that D and E can move depending on ΔABD and ΔACE; (e) realize that AE=DF and AD=EF are nontrivial; (f) attend to AB=AD and AE=AC; (g) measure CE, AC, AB, BD,BC, BF and CF; (h) be aware of DB=AB; (i) seeing a special case, notice that all the sides except the sides of ΔFBC are of the same length; (j) notice that ΔEFC, ΔDBF, and ΔABC were isosceles triangles, and the sides of AEFD are the sides of isosceles or equilateral triangles; (k) notice that the forms of ΔEFC and ΔDBF depends on that of ΔABC; (l) find that the opposite sides were of the same lengths because of some isosceles and equilateral triangles; (m) seeing the breaking down of a special case, find that ΔEFC, ΔDBF, and ΔABC are congruent; (n) notice that rotating leads ΔABC to ΔDBF and ΔEFC; (o) dragging test of the congruence of ΔEFC, ΔDBF, and ΔABC; (p) understand why EF is of the same length as AD when assuming the congruence of ΔEFC and ΔDBF.

Although this pair could not complete their proof, they could find the congruence of ΔEFC, ΔDBF, and ΔABC, which is critical for the proof of their conjecture. Like another pair, they deepened their understanding of the situation gradually; especially, they deepened their understanding of the dependency of ΔEFC and ΔDBF on ΔABC. They also explained some relationships of the sides here and there by resorting to the given equilateral triangles, and checked relationships between the sides of the parallelogram ADFE and the sides of these triangles. As their understanding deepened, the students noticed that the relationship of DF and EF to other sides may be critical. Observing the special case, they came to consider DF and EF as the sides of ΔDBF and ΔEFC respectively.

DGS helped the students explore the dependency of the triangles through its dragging and measuring functions. Like the use of DGS by another pair, a certain special case and its slight break played an important role in the proving process of Pair B. When the students made ΔABC an isosceles triangle, they noticed that ΔEFC and ΔDBF also became isosceles. Breaking down this special case, they realized the dependency of ΔEFC and ΔDBF on ΔABC. And after the second breaking down of this special case, Nogawa could find that ΔEFC, ΔDBF, and ΔABC were congruent. While their construction of the special case and the first breaking down were implemented intentionally, the second breaking down happened by chance and were beyond the students' intentions. This means that an emergent pattern which had appeared by chance played a critical role also in Pair B's proving process.

Just before they noticed that ΔEFC, ΔDBF, and ΔABC were isosceles triangles, they observed the measurements on the screen and told that while they understood why the lengths of AD and AE became 8.59cm, they had no idea about why the lengths of DF and EF also became 8.59. Their attention to these sides might help them realize that all the three triangles were isosceles. Before the students noticed the congruence of ΔEFC, ΔDBF, and ΔABC, Yama explained why the opposite sides were of the same lengths when ΔABC was an isosceles triangle and they paid attention to these three triangles. Such understandings of the situation, as well as the measurements on the screen, facilitated the students' recognition and interpretation of the emergent patterns on the screen and enabled them to find the important relationships in the situaiton.

At the beginning of their activity, Nogawa and Yama dragged the point A so that it lay near the mid of BC and moved A upward. They did such dragging a few times. Thus, the event similar to Figure 3 happened for them. However, they noticed nothing about ΔEFC and ΔDBF. They paid attention only to the quadrilateral ADFE and could not yet develop their understanding of the situation. It may be said that without a certain level of understanding of the situation, solvers cannot take advantage of the events which occurred on the screen.

4. Discussion

The above analyses illustrates that a kind of interactions can be found in the students' proving processes, in which the students' explorations led to obtaining or being aware of the information about the problem situation and their understanding deepened by that information triggered next explorations and supported their interpretations of the figures which appeared during those explorations. Even the critical idea for the final proof was realized through such a gradual deepening of the students' understanding. This implies that deepening of solvers' understanding of the problem situations can be considered an important aspect of their proving processes.

If taking this standpoint, it should be paid attention to how solvers' understandings of problem situations are deepened gradually through their explorations. The above analyses focusing on such an aspect shows, for example, that slight deviations from special cases can play an important role for finding new information about situations, as well as the dragging preserving certain properties or lieu muet dragging (Arzarello et al, 1998; see also Leung & Lopez-Real, 2002). Manipulating figures and "getting in touch with the problem" (Furinghetti et al., 2005) is useful not only at the early stage of proving, but throughout proving processes.

It should be also paid attention to how solvers' explorations are supported by their understanding of problem situations. Hollebrands (2007) pointed out that solvers' interpretations of what are happening on the screen are influenced by their mathematical knowledge. The above analysis suggests that their interpretations, as well as their use of DGS, are influenced by their understanding of the problem situations. While measurement functions as a means for finding explanations (Christou et al., 2004), what to be measured and what or how to drag is also influenced by their understanding.

In the previous researches, much attention has been paid to transitions from spatio-graphical reasoning to theoretical-geometrical reasoning or moves between them (Olivero & Robutti, 2007). The analyses of this manuscript suggest that deepening of solvers' understanding and its relationship with their explorations can be another dimension for our analyzing and understanding students' proving processes. Since it does not necessarily require the distinction between spatio-graphical and theoretical-geometrical, this dimension will be applied to proving in the areas other than geometry.

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