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Chapter 7

Elementary School Students' Use of Drawings and Their Problem Solving

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Abstract

It has been said that students' self-generated drawings are helpful for their mathematical problem solving. It is also known, however, that it is not easy for students to construct appropriate drawings for solving the problems at hand. Students need to understand the problems enough to construct those helpful drawings.

To resolve this contradiction, Nunokawa (2006) proposed a framework in which solvers' use of drawings and their understanding of problem situations deepen each other. Because the validity of this framework was demonstrated only for adult problem solvers,it is still unclear whether this framework can be applied toproblem solving processes of younger students.

In this chapter, problem solving processes of 5th grade students will be analyzed focusing on their use of drawings and on how those drawings helped their problem solving. The first analysis will deal with the problem solving process of one pair of students. To solve a little difficult mathematical problem, those students made several drawings and could solve the problem successfully. The analysis will demonstrate that the students represented what they noticed about the problem situation at each step in their drawings and obtained new information about thesituationfrom those drawings even thoughmost drawings did not provide them with direct clues for solutions. This will imply that elementary school children's use of drawings can be investigated through the framework.

In the second analysis, the solving process in which a teacher intervened will be analyzed. The teacher's intervention was based on the framework and encouraged a student to represent what she had in mind at each step and obtain new information about the problem situation. The analysis will show that through making several drawings, the

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student changed her understanding of the problem situation from superficial one to deeper one which reflected the problem structure.

The results of these analyses will show that the framework is also useful for investigating problem solving processes of younger students and for designing teachers' supports for those students' problem solving.

Keywords: mathematical problem solving; Heuristic strategies; drawings; problem solving processes; understanding of problem situations

INTRODUCTION

The use of drawings or diagramsis a well-known problem solving strategies in mathematics education (*e.g.* Schoenfeld, 1985). They play some roles like the following (Van Essen andHamaker, 1990): (1) drawings relieve working memory: (2) by generating drawings, solvers make problems more concrete; (3) problem information can be reorganized in drawings so that solvers can look for related information more efficiently (Larkin and Simon, 1987); (4) some problem characteristics are more easily inferred from drawings,orrelationships between the elements in the problem can be more explicit in drawings (Gutsteinand Romberg, 1995). It is often said, however, that while expert solvers can choose appropriate drawings and use them effectively, it is very difficult for novice solvers or students to do so. This fact has led some researchers to recommending that students should learn what kinds of representations are useful for certain types of mathematical problems (*e.g.* Charles *et al.*, 1985; Uesaka and Malano, 2006; Van Garderen, 2007; Veloo, 1996). Furthermore, the drawings or diagrams used in mathematical problem solving need to be schematic, not pictorial, in that they represent structural relations rather than specific details (Hegarty and Kozhevnikov, 1999). This necessity can be supported by the fact that people who are good at mathematics, science or technology tend to use schematic images in their reasoning (BlazhenkovaandKozhevnikov, 2009). Based on such researches, it is also said that students should be taught to construct schematic diagrams in solving mathematical problems (*e.g.* Edens and Potter, 2008; Güler and Çiltaş, 2011).

In recent discussions about the use of drawings, there are some researchers who have attended to the fact that drawings are usually modified or altered during problem solving processes (Bremigan, 2005; Nunokawa, 2000). For example, Gibson (1998) analyzed the problem solving processes of university students to investigate the effect of drawings on developing proofs. He observed that in their search for proofs, "students added to, subtracted from, and redrew" their drawings and their alteration of drawings "allowed students to explore possible scenarios within the problem situation and, as a result, come up with ideas" (p. 296). In their study of solution processes of high- and low-achieving secondary students, Lawson and Chinnappan(1994) considered the modification of drawings to be a kind of information generation, and they paid attention to the orders in which the elements of drawings wereadded. Diezmann(2000) pointed out that diagrams were dynamic rather than static representation and it could be beneficial to produce more than one diagram.

Such a viewpoint may imply the following ideas: It is not necessary for a solver to construct a very appropriate drawing which can directly lead him/her to solutions; Even when a solver constructs a rather naïve drawing at the outset, it may be possible to make it better one in his/her problem solving process. Some researchers have investigated how drawings are actually used in mathematical problem solvinganddemonstrated the interrelationships between uses of drawings and analytical thinking (Hershkowitz et al, 2001; Nunokawa, 1994; Presmeg and Balderas-Cañas, 2001; Stylianou, 2002; Zazkis et al, 1996). They have highlighted solvers' analytical operations on drawings, and some of them emphasized the complementary relationship between drawings and analytical thinking.

Furthermore, Nunokawa (2004, 2006) analyzed the problem solving processes where graduate students of mathematics education solved difficult plane-geometry problems and he found that while the solvers constructed well-designed drawings using the information they had obtained in previous phases of their problem solving processes, drawings contributed to the generation of new information by producing unexpected combinations of the elements in problem situations and prompting the solvers to recognize emergent patterns.

Consequently, he presented the framework about uses of drawings which includedan interactive process in which solvers' understanding of problem situations and their uses of drawings revise each other as follows: (1) Solvers represent what they know about problem situations at that time; (2) Drawings present new information and solvers improve their understandings of problem situations; (3) Solvers modify their drawings or construct new drawings to incorporate new information into their drawings. (See also the schematic expression of this process in Nunokawa (2010)).This framework about uses of drawings in which solvers elaborate their drawings gradually seems to provide us with a perspective for analyzing how solvers use drawings and how drawings can be useful for mathematical problem solving.

It was developed, however, from the analyses of graduate students' solving processes. Therefore, it is still unclear whether similar processes are observed in problem solving processes of younger students and whether it can be useful for supporting younger students solving mathematics problems. The purpose of this chapter is to examine these issues.

1. METHOD

For this purpose, two problem-solving processes from the sessions recorded in our research will be analyzed in detail. In these problem-solving sessions, 5th grade students were required to solve mathematical problems.

The problems which were a little difficult for 5th grade students were chosen in order to explore the roles which drawings play in their problem solving: (a) How elementary school students use drawings when they confront difficulties in their problem solving; (b) How teachers' interventions in relation to drawings (*e.g.* "How about making a drawing about this problem?") help those students.

The interviewer, the second author, did not intervene in students' problem solving until their solving activities stagnated or students were satisfied with incorrect solutions. When the interviewer intervened, he suggested making a drawing about the problem situations. Based on the framework discussed in the previous section, naïve drawings (e.g. pictures or images of the problem situations) were accepted as the first step.

When students noticed something about the problem situations through making drawings, the interviewer encouraged them to represent it by modifying the drawings or making new drawings.

The students solved the problems individually or in pairs. We asked the classroom teachers to choose students who tended to speak well about their thinking so that their solving processes would be observable.

The teachers selected the students, and made pairs of the students or allowed some of them to solve individually when they wished to do so. Two video-cameras and one audio recorder were used to record the students' problem solving activities.

One of the cameras was used to record the students' worksheets and what and when they wrote, and another camera was used to record the students' upper bodies including their facial expressions. The video-recorded data were transcribed into protocols which included what and in what order the students wrote on their worksheets as well as their utterances. These protocols and the students' worksheets were the data analyzed in this study.The data were analyzed qualitatively and, in these analyses,the focus was laid upon how the students used drawings in their problem solving.

The data used in this chapter are a problem solving process of a pair of two students and that of one student. Because this pair of students could solve the problem without help of the interviewer, we will treat their solving process as a non-interventional session and, through the analysis of this session, obtain insights into how successful elementary school students use drawings for solving the problem.

Because the student needed much help from the interviewer in the latter case and the interviewer suggested making drawings several times, we will treat her solving process as an interventional session and, through the analysis of this session, obtain insights into how teachers' help based on the above-mentioned framework can promote elementary school students' problem solving. According to the classroom teachers, two students who participated in the non-interventional session were above average and at average level in their class respectively, while the student who participated in the interventional session was at average level.

2. STUDENTS' USE OF DRAWINGS IN SOLVING MATHEMATICAL PROBLEMS

In this section, the problem solving process in the non-interventional session will be analyzed to explore how successful students used drawings to promote their problem solving.

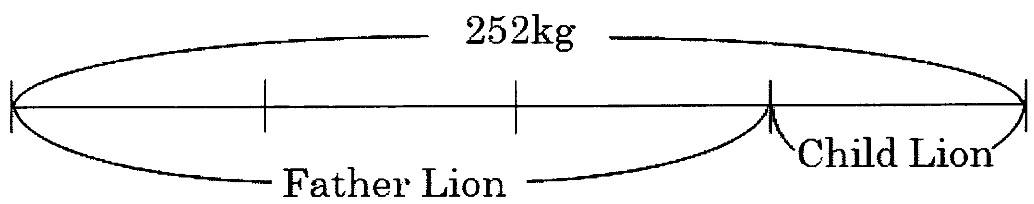

Figure 1.

2.1.TheStudents' Use of Drawings andTheir Solving Process

The pair of students, Shiho and Fumi (5th grade girls, pseudonyms), tackled the following problem in their problem solving session.

Problem: There are a father lion and a child lion. The sum of the weights of these two lions is 252kg and the father lion weighs three times as much as the child lion. What are the weights of the father lion and the child lion respectively?

This problem is a little difficult for 5th graders (Kikuchi, 1996). Therefore, as mentioned in the previous section, it was expected that students would show extended problem solving processes and use some drawings during those processes to solve this problem.

The problem can be solved, for example, by using the following line diagram and calculating 252÷4=63 and 63x3=189. Shiho and Fumi finally constructed the drawing which was very similar to this line diagram and calculated 252÷4=63 and 63x3=189. But they also constructed several other drawings which could not directly lead them to their solution and most of which were more pictorial ones.

2.1.1. First Phase of Their Problem Solving (about 2 Minutes)

After reading the problem statement, Shiho said, "Let's start with a father lion and a child lion" and "Three times of this, so...," and she drew the drawing shown in Figure 2 which included the information of "252kg" and "x3" in a superficial way.Seeing this Shiho's drawing, Fumi said, "What? What are you doing? Why do you calculate 252x3?" Shiho was not calculating 252x3, but she told Fumi that the problem said "three times." When Fumi emphasized that 252 is the sum of the weights of two lions, Shiho told that she came to have no idea and looked confused.

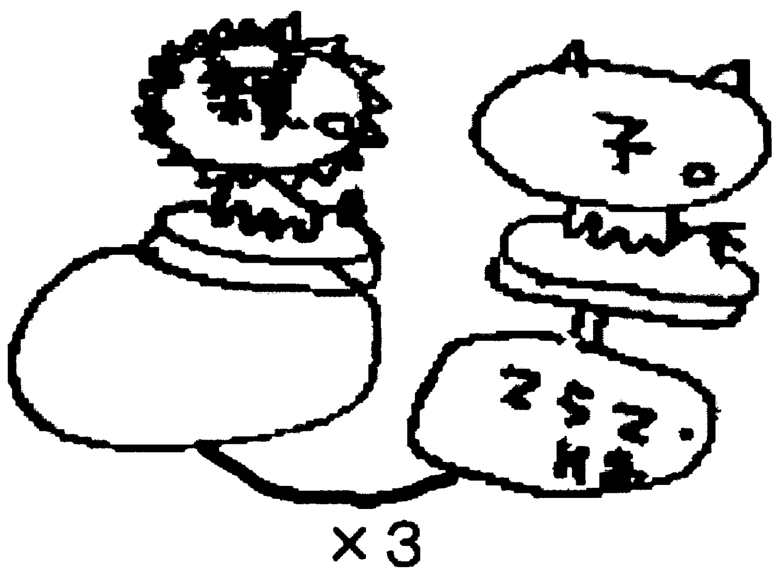

Figure 2.Two Chinese characters in the faces mean "father" (left) and "child" (right) respectively.

2.1.2. Second Phase of Their Problem Solving (about 1 and a Half Minutes)

Fumi said, "OK, Let's divide it by 2," and began to calculate 252÷2 to find 126 as the quotient. Shiho first resisted this strategy saying, "Divide?...But, well, well, we need to make it three times more, so if we halve it..." But she soon said, "Well, all right, I'll do it after halving," and began to calculate 252÷2, too. Shiho called the quotient 126 "a tentative answer" and drew a drawing in Figure 3. After drawing it, Shiho said, "this [left 126 in Figure 3] is as three times much as this [right 126], so this one [left 126] is...Hmm..." and looked confused again. Fumi calculated 126÷3=42 and 252-42=210. She wrote 210 and 42 as tentative answers, which she thought were the weights of the father lion and the child lion respectively.

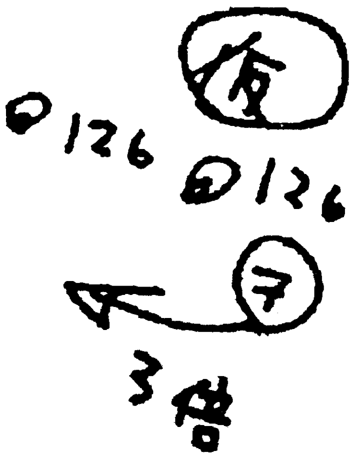

Figure 3.Chinese characters mean "tentative" (top), "child" (middle), and "3 times" (bottom).

Figure 4.Japanese character at the top expresses a roar of the lions; Japanese characters in the faces in the left and right lions are the initials of "father" and "child" in Japanese.

2.1.3. Third Phase of Their Problem Solving (about 1 and a Half Minutes)

Next, Shiho said, "Adding this and this should make 252, shouldn't it?" and drew a new drawing (Figure 4) which represented only the total of the weights of the father and the child lions. Two students checked whether 42x3 became 210 and found that these answers did not satisfy the multiplicative condition given in the problem. Shiho uttered here as follows: "Thus, we need to find the triple of...Uh-oh"; "Make the triple of...well"; "It's confusing."

2.1.4. Fourth Phase of Their Problem Solving (about 13 and a Half Minutes)

Then, Shiho temporarily shelved the condition about the total weight and focused on another condition, three times. She drew a new drawing (Figure 5) in the following order: Drew a child lion (right) and wrote "3 x" saying, "the triple of this is..."; Drew a father lion (left) saying, "Its triple is this, isn't' it"; Deleting the initial "3" with a double line and rewrote another "3"; Said, "The original one there was before tripling is the child, so..."; Wrote "+" and "252" below the lions saying, "the total is 252." Furthermore, Shiho added two children to Figure 5 to make Figure 6, talking as follows: "If there are three children, it's a little strange, but if there are three children, Hmm, if there are three children, they will be balanced with the weight of the father."

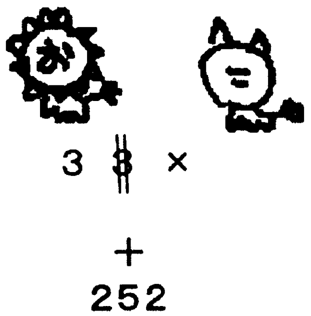

Figure 5.

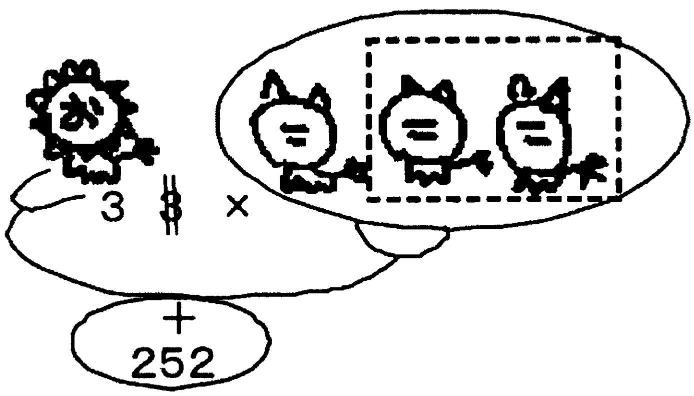

Figure 6.

Seeing Figure 6, Shiho mentioned the total and 252 several times and began to calculate 252÷2. She found out the quotient to be 125 [sic] and wrote "tentative 125." She calculated 125÷3 to find 41.6 saying, "Am I doing a wrong thing? No, it may be right," and calculated

41.6x3. Shiho said to Fumi, "I've halved it." After that, Shiho seemed to come up with something saying, "Ah, Ah, Ah."

She immediately began to calculate 252–41.6=211.6 [sic] and 211.6–41.6=170. Fumi pointed out the error in Shiho's calculation and they found out that the quotient of 252÷2 is 126. Shiho calculated 126÷3=42 and told that she obtained the exact answer and 42kg is the weight of one child. They also told that the weight of the father lion had to be as 3 times much as this 42. But they looked a little confused and Fumi said, "What are we doing?"

They confirmed that the total weight is the sum of the weights of ONE father lion and ONE child lion. Shiho said, "The weight of ONE child lion... It might be wrong that I added children earlier." She also said, "Because it weighs as 3 times much as the child, Ah, [looking at Figure 6] this is the father...Ah, Ah, I need to divide its weight into three, don't I?" While Fumi was calculating 252÷3=84, Shiho looked at her worksheet and said, "Oh No, my calculations may be totally wrong, Oh well, I'd better go back to the beginning and start over."

When Fumi told Shiho that she calculated 252÷3 and found 84, Shiho said, "What? Wait! Maybe that is already a wrong way." Shiho said, "Ah, Ah," and began to calculate 252÷4. When Fumi asked why Shiho chose "4" as the divisor, Shiho told that she chose it somehow and could not explain a clear reason. Shiho calculated 63x3=189. Fumi asked again where the divisor came from, but Shiho simply explained what she calculated. When Fumi seemed unsatisfied with her explanation, Shiho calculated 252÷4 again and calculated 252–63=189 this time. Seeing the results of these calculations, Shiho considered 189 and 63 the answers which were the weights of the father and the child lions. When Fumi calculated 189+63=252 and 63x3=189 with Shiho, Fumi accepted these answers and asked Shiho how she found them.

2.1.5. Fifth Phase of Their Problem Solving (about 6 Minutes)

When Shiho began her explanation saying, "First, I divided it by 4,"Fumi asked where that "4" came from. Then, Shiho constructed a drawing shown in Figure 7 for her explanation as follows: Drew a right ellipse saying, "Three children"; Drew a left circle saying, "One father"; Added two vertical lines to the ellipse, saying, "Because [it weighs] three times, you know,... So, it's equivalent to 3 children"; Added a small bullet to the circle and 3 bullets to the ellipse saying, "I assumed those one, two, three, four lions, the same lions"; Added the two horizontal lines at the top and the bottom of the drawing. After that, Shiho also told that she could find the answers even though she did not know well how she had found them. When Fumi read the problem statement and highlighted that there were one father and one child and that what matters was the triple of the weight of a child lion, Shiho said, "Therefore, one weighs 63." Shiho made another drawing (Figure 8) saying as follows: Drew the right ellipse saying "Then there is one lion"; Drew the left ellipse saying, "and there is this one, too"; Added small circle in the right ellipse saying, "First, I normally thought that tripling means three lions"; Added two lines like a form of long division saying, "so I divide this by 3"; Drew the ellipse above the line (i.e. the position of quotients) saying, "Thus, I've divided it into four like this"; Added the arrow to this ellipse saying, "Then I triple this one."

When Fumi asked again where the divisor "4" came from, Shiho said as follows: "4, so 4 came from, well..., before tripling it, the weight of the father is equal to that of one child [sic], but, when I divided it by 2, tripling led to the very different weight, therefore I changed to dividing it by 4."Fumi was not satisfied with this explanation. Shiho added the following comment: "Oh, but I really obtained these numbers by dividing it by 4."

Fumi told that she did not understand Shiho's idea and asked whether it was appropriate to divide the father into three children. Shiho answered "yes" to Fumi's question and drew the father lion of Figure 10. Furthermore, Shiho drew another drawing (Figure 9) and said, "Ah, what I wanted to say is considering this 'three lions' as the father and considering this one [left circle] as the child."Fumi asked, "Three of what?" Shiho answered, "Three times of the child's weight," and added the child lion of Figure 10. Fumi told this time that she might understand Shiho's idea.

Fumi drew three same Chinese characters in the left ellipse in Figure 11 saying, "Child, Child, Child, there are three children." Shiho added an ellipse around these Chinese characters saying, "And we consider this as the father."Fumi added "+" and the right Chinese character to this drawing saying, "Adding up three children makes the father."

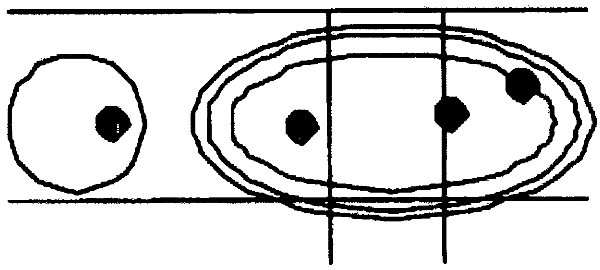

Figure 7.

Figure 8.

Then, Shiho explained as follows: "If we consider this 'three children' as the father and consider this remaining one as the child, dividing it by 4 leads to those answers.

As we deal with those answers in the same way as this [Figure 11], we multiply it by 3 and its product becomes the father's one, and the remaining is this one, then we can get this." Shiho reported to the interviewer that they got to the above-mentioned answers and Fumi wrote as the answers, "The father lion 189kg, the child lion 63kg." Shiho drew a drawing in

Figure 12 saying, "We assumed three children and considered the set of them to be the father." She drew one more drawing similar to Figure 12 saying, "We found out the weight of one child and separated them [the father and the child] into three children and one child, and then we obtained those answers."

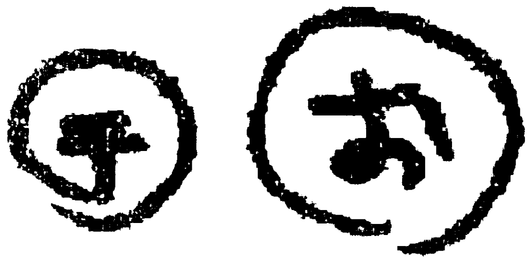

Figure 9.Character in the left circle means "child" and that in the right circle is the initial of "father."

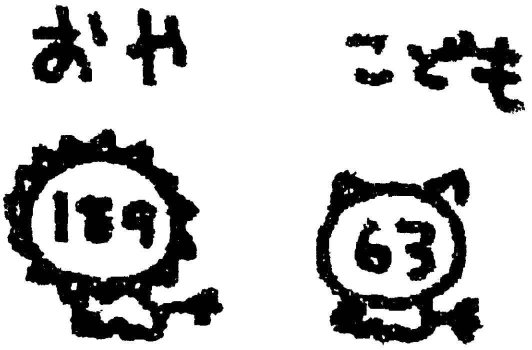

Figure 10.Japanese characters above the left and right lions mean"father"and"child" respectively.

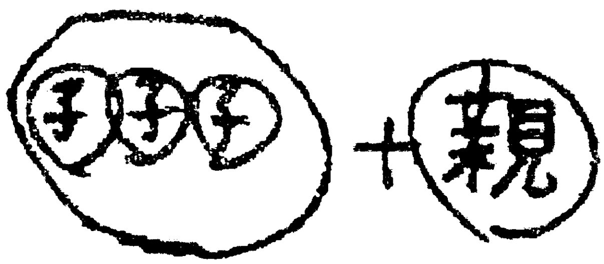

Figure 11.Chinese characters in the left circles mean "child" and the letter in the right circle means "father."

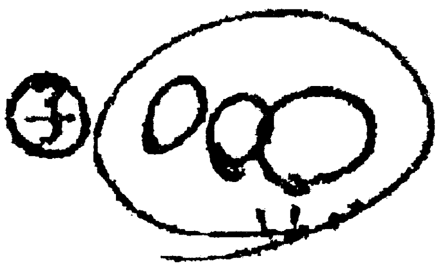

Figure 12.Chinese character in the left circle means "child."

2.1.6. Post-Interview with Shiho and Fumi (about 5 Minutes)

When the students were asked in the post-interview what they calculated in the earlier stage of their problem solving process, Shiho told that their earlier calculation, dividing 252 by 2, was totally irrelevant and said, "We divided the total weight by 2 because we thought that the father and his child are of the same size." When she was asked how she noticed the divisor "4," Shiho pointed at Figure 6 but could not explain it well as follows: "I assumed that there are two more lions behind the child lion and made it triple, there is also one father, I supposed there are three fathers, too, and I thought the child is bringing along two lions, so...." She explained further as follows: "First, I thought a set of three children was equivalent to the father, but next I came to think the father was equivalent to three children, and all of them had the same value, that led me to the answers." When Fumi told here that she could not understand Shiho's explanation well, Shiho explained her idea again making drawings: "[making a drawing similar to Figure 11] Well, first, I couldn't solve the problem because I thought three children and one father, [making a drawing similar to Figure 12] but later I came to think the father is equivalent to three children and there is only one child, then I could solve it."

2.2. Characteristics of the Students' Use of Drawings in Their Problem Solving

2.2.1. Drawings as an Origin of the Main Idea

When she explained her idea, Shiho used the drawings (Figure 7, Figure 12) which were similar to the line diagram of the problem structure (Figure 1). Although those drawings did not include the numerical information about the total weight, 252, they represented the relationship between the weights of the father and the child and showed that the total is the combination of four equivalent weights. These drawings implied the reason why the division 252÷4 could lead to the weight of one child.

This critical element, the divisor 4, was found by Shiho in a slightly different drawing, Figure 6. Because this drawing was drawn to make explicit the relationship that the weight of the father lion is the triple of the child's weight, it included one father and three children instead of one child and three children. The drawing of one father and three children resulted in the appearance of four lions, but this appearance was not expected by Shiho in advance and it occurred without her intention. That it, the information of "4" is an emergent element (Nunokawa, 2006; Roskos-Ewoldsen, 1993; Sawyer, 2003) in this drawing. In fact, Shiho did not notice this element when she drew it and began to calculate 252÷2. She noticed the idea of dividing the father's weight into three only after Shiho and Fumi calculated 252÷2 and the related calculations and found that this line of thought could not be successful. At that time, Shiho realized that the set of three children in Figure 6 represented the father and changed her idea from adding two more children to dividing the father into three ("Ah, this is the father...Ah, Ah, I need to divide its weight into three, don't I?"). Reexamining and reinterpreting the drawing, which was not a fully appropriate representation of the problem situation, made it possible for Shiho to find the critical element for solving the problem.

This process of finding the critical element can be supported by Shiho's explanation in the post-interview: "[making a drawing similar to Figure 11] Well, first, I couldn't solve the problem because I thought three children and one father, [making a drawing similar to Figure 12] but later I came to think the father is equivalent to three children and there is only one child, then I could solve it." Although the drawing in Figure 6 was not a representation of her sufficient understanding of the problem structure, it could help Shiho find the critical element and construct the appropriate numerical expressions.

2.2.2. Gradual Modification of Drawings

As discussed above, while the drawing similar to the line diagram (Figure 12) was constructed at the later stage of their problem solving processes, the drawing which was most helpful for their problem solving (Figure 6) was not a representation of Shiho's sufficient understanding of the problem situation. Reexamination and reinterpretation of it led Shiho to finding the critical element.

Furthermore, this drawing was a consequence of a series of drawings some of which were more pictorial ones. Figure 6 first started as Figure 5 which is a variation of Figure 4. While the drawing in Figure 4 represented one condition that the sum of the weights of two lions is 252kg, Shiho tried to add another condition, the father lion weighs three times as much as the child lion, to this drawing and wrote "3 x" in Figure 5. When she reinterpreted this second condition as the condition that one father is equivalent to three child lions, Shiho added two more child lions to Figure 5 and modified Figure 5 into Figure 6. In the process of constructing Figure 5 by elaborating Figure 4, the picture of a weight scale, on which two lions stood in Figure 4, had been omitted.

Although the drawing in Figure 4 was far from the schematic representation of the problem structure because it represented only one condition and included unnecessary concrete elements (a weight scale, detailed picture of the lions, and a roar of the lions), this drawing seemed to play an important role in the students' problem solving. Before Shiho drew this drawing, the students calculated 252÷2=126 and 126÷3=42. Drawing Figure 4 saying, "

Adding this and this makes 252," directed Shiho's attention to the additive relation between the weights of the father and child lions. The students began to check whether the results of 42x3 and of 252-42 became the same and found that this line of thought was inappropriate.

Moreover, this checking activity enabled Shiho to notice that the number to be tripled or the tripled number was the key element of this problem ("Thus, we need to find the triple of...Uh-oh"). That is, if the number to be tripled or the tripled number can be found, all the elements in the problem situation can be derived from it. This idea was reflected in her final solution.

Figure 4 did not represent the condition that the father lion weighs three times as much as the child lion, but the father lion in Figure 4 was bigger than the child lion.

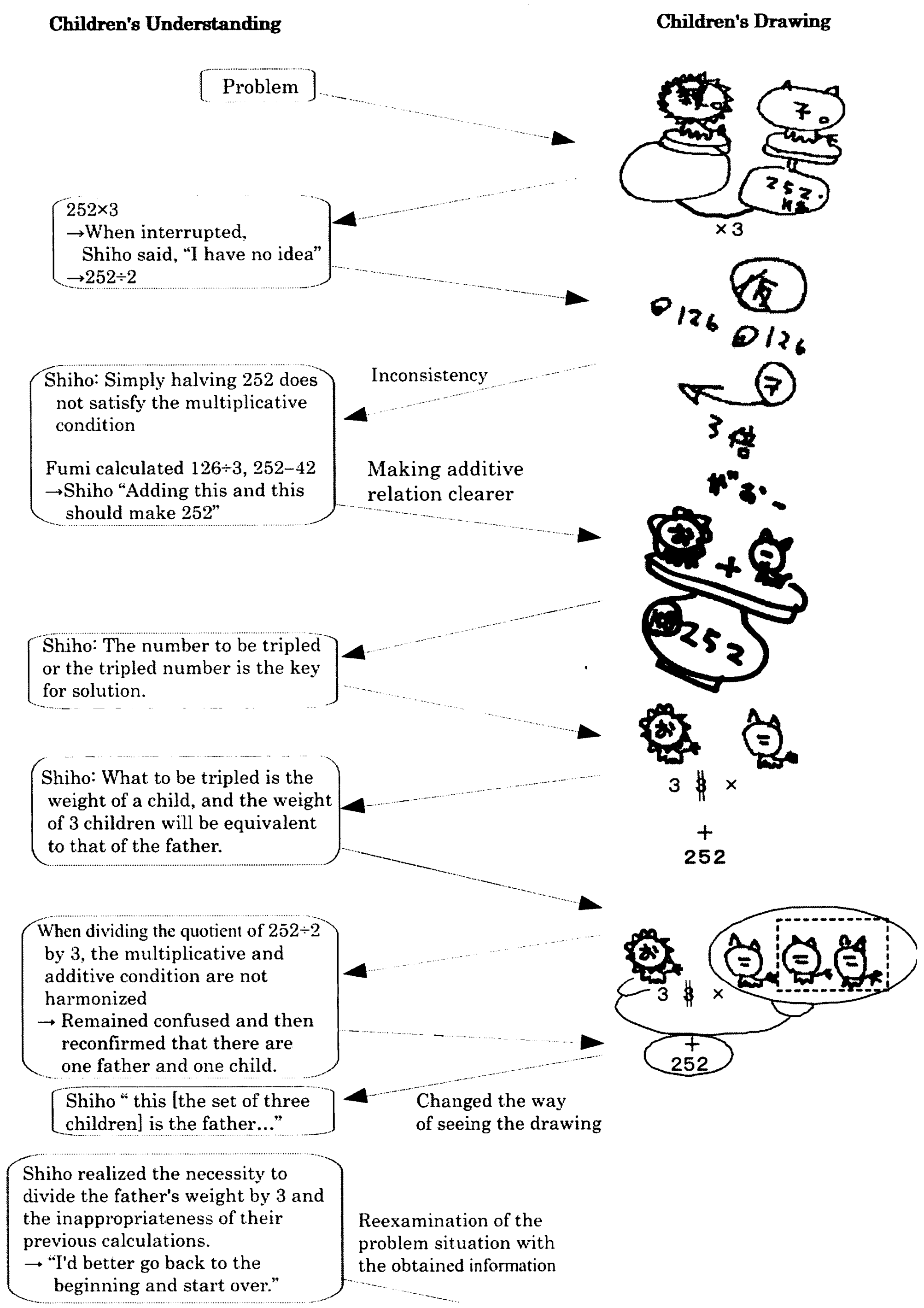

Figure 13. (Continued).

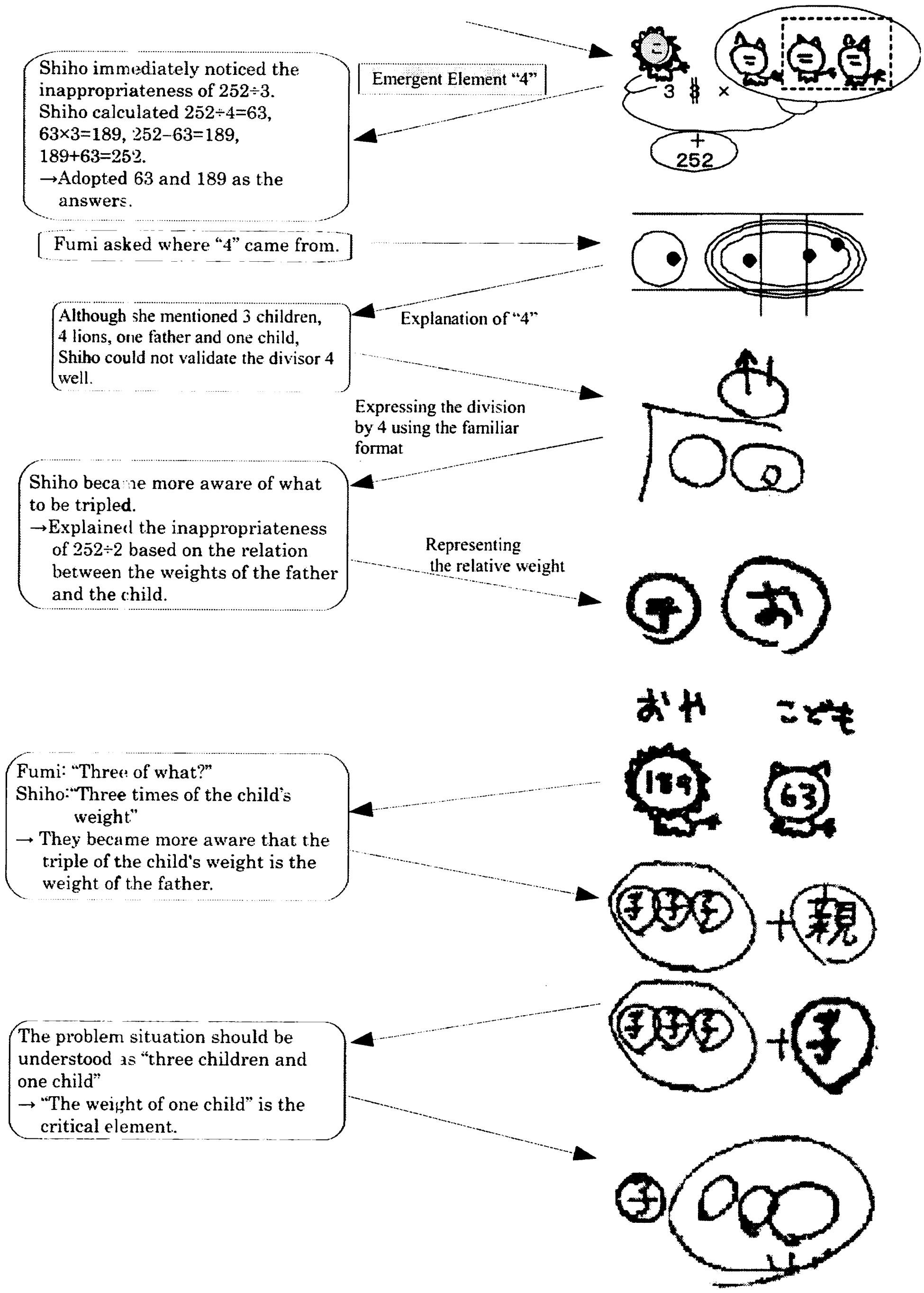

Figure 13.

According to the students' response in the post-interview, they simply thought at the beginning that "the father and his child are of the same size." Shiho represented the result derived from this assumption as "a tentative answer" in Figure 3 and checked it as follows: "this [left 126 in Figure 3] is as three times much as this [right 126], so this one [left 126] is...Hmm..." Figure 3 enabled them to examine the consequence of their (incorrect) assumption and become more aware of the condition that the father lion is bigger than his child. This examination was facilitated by the feature of Figure 3 that represented what to be tripled explicitly with an arrow, which Figure 2 did not have.Figure 3 was the representation of the information the students had obtained referring to Figure 2. In the drawing in Figure 2, the child lion is bigger than the father lion and the total weight is written under the child lion. Even though the condition "x3" was included in this drawing, which weight should be tripled was unclear in it. This feature of the drawing led the student to the incorrect assumption that "the father and his child are of the same size." The flow of their use of drawings is summarized in Figure 13. It shows the following flow: (a) Their use of drawings and their understanding of the problem situation changed interactively; (b) In this interaction, their drawings changed from more pictorial ones toward more schematic ones. In other words, even pictorial drawings functioned as steppingstones of their problem solving process. Those drawings represented the students' understanding at that time, helped them explore the problem situation taking account of that understanding, and provided them with new pieces of information about the problem situation even though those drawings were not directly useful for the final solution. In this sense, the elementary school students' use of drawings analyzed in this section is consistent with the framework presented in Section 1.

2.2.3. Time Lags of Understanding of the Problem Situation

Even though Shiho found the critical element, the divisor "4," when she reinterpreted the drawing in Figure 6, she could not clearly explain the reason why she divided the total weight by 4. Shiho told that she chose that method "somehow" and simply repeated the related calculations. But, at the end of their problem solving process, Shiho was aware of what this "4" meant well and could explain that this "4" came from the idea that the problem situation can be understood as the combination of three children and one child.

Shiho: "We assumed three children and considered the set of them to be the father."

Shiho: "We found out the weight of one child and separated them [the father and the child] into three children and one child, and then we obtained those answers."

As the latter part of Figure 13 shows, the students' understanding of the problem situation changed gradually even after they noticed the critical element and found the calculations necessary for the final solution. In this later part, their understanding of the problem situation interacted with their use of drawings and they became clearer hand-in-hand. During this interaction, the students came to distinguish two similar drawings more clearly: One type of them (Figure 11) represented that a set of three children is equivalent to one father, i.e. the multiplicative condition: Another type of them (Figure 12) represented a combination of one child and a set of three children as one father, i.e. an integration of the additive and the multiplicative conditions of this problem. This distinction led to their deeper understanding of the problem situation and clearer reason of the validity of the divisor "4."

Shiho noticed that there are four "same lions" in the problem situation when she drew Figure 7, but she could not distinguish two types of drawings well at that stage and told "three children" and "one father." This confusion was indirectly reflected in Figure 8. The three ellipses in Figure 8 were of the same size, although the right one represented the father and the left and the upper ones represented one child.

The small circle in the right ellipse was very smaller than the left and the upper ellipses, although all of them represented one child. Shiho could not clearly associate the divisor "4" with the four lions and said."Oh, but I really obtained these numbers by dividing it by

4."While the drawing in Figure 9 did not include the multiplicative relation, the relative size of the father and the child was represented clearly in it. Based on this relative-size relation, Shiho could clearly explain what three lions and another lion represented respectively: "What I wanted to say is considering this 'three lions' as the father and considering this one [left circle in Figure 9] as the child."

The combination of Figure 9 and her explanation was an integration of the additive and the multiplicative conditions. When Fumi drew Figure 11 to represent her understanding, Shiho could explain how to reinterpret that drawing to validate the divisor "4": "If we consider this 'three children' as the father and consider this remaining one as the child, dividing it by 4 leads to those answers." This was what she represented in Figure 12. At this final stage, Shiho eventually became fully aware of how she had reinterpreted Figure 6 to find the divisor "4."

This analysis of the latter part of their problem solving process shows that even when the students found the calculations necessary for the final solution, they did not necessarily understand the problem situation enough and their understanding of the situation could deepen after that. There was a temporal difference between their insight into the solution and their full understanding of the situation.

Furthermore, the analysis also shows that in the process of making clearer their understanding of the problem situation, the interaction of their understanding and their use of drawings continued. In this sense, the framework about students' use of drawings presented in Section 1 can be applied to the whole process of their problem solving.

3. Student's Problem Solving with Teacher's Intervention Based on the Framework

In this section, the problem solving process in the interventional session will be analyzed to explore how a teacher's encouragement to make drawings can be helpful for a 5th grade student. Because, as discussed in the previous section, the framework about students' use of drawings can be applied to elementary school students' problem solving processes, the teacher's support in this interventional session was provided based on this framework.

3.1. The student's Use of Drawings and Her Solving Process

The student Maki (a 5^{th} grade girl, pseudonym) tackled the following problem in her problem solving session.

Problem: Two children, Kiyoshi and Akiko, want to make paper cranes with 60 square papers called Origami in Japan. To make those cranes, they are dividing those 60 papers so that Akiko will have 12 more papers than Kiyoshi. How many papers will each student have?

This problem was selected for this student because the classroom teacher thought that she was not good at learning mathematics (See her initial answer in Figure 15). This problem has the feature similar to the Lion problem: the total of and the relationship between two objects are known and students are required to find the quantity of each object. While the given relationship between two objects is multiplicative in the Lion problem, the relationship is additive in this Origami problem. In this sense, we considered the Origami problem easier than the Lion problem.

The structure of this problem can be represented, for example, as the following diagram, and the problem can be solved in two manners:

Solution A: 60–12=48, 48÷2=24, 24+12=36
Solution B: 60÷2=30, 30–6=24, 30+6=36.

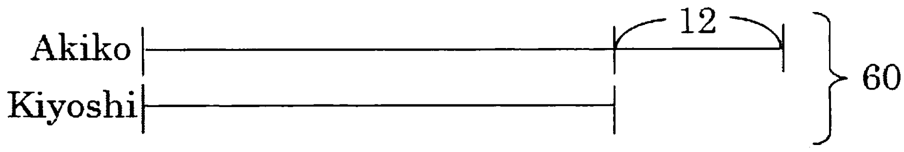

Figure 14.

It is very difficult for 5th grade students, especially for students other than high-achievers, to make such a highly structured diagram. With the interviewer's supports, Maki started with a pictorial drawing and changed it gradually to reflect the problem structure more appropriately.

3.1.1. First Phase of Her Problem Solving (about 13 Minutes)

After reading the problem statement, Maki wrote "Expression 60÷12=" and implemented this division to find the quotient 5 (Figure 15). She said "Uh-oh" during this calculation, indicating that she was a little puzzled.

When the interviewer asked her to explain her solution, Maki explained as follows: "This '60' means 60 square papers and '12' [of 60÷12] is the '12' of the '12 more papers' in the problem statement. '5' is the answer of this calculation." When the interviewer asked why she had said "Uh-oh" and looked puzzled, Maki told as follows: "First I thought each child got 30 papers because they divided 60 papers. But the statement says Akiko will have 12 more. So, I was not sure which I should write in this place[indicating the divisor of 60÷12], 2 or 12." She said that she was still a little puzzled at that time.

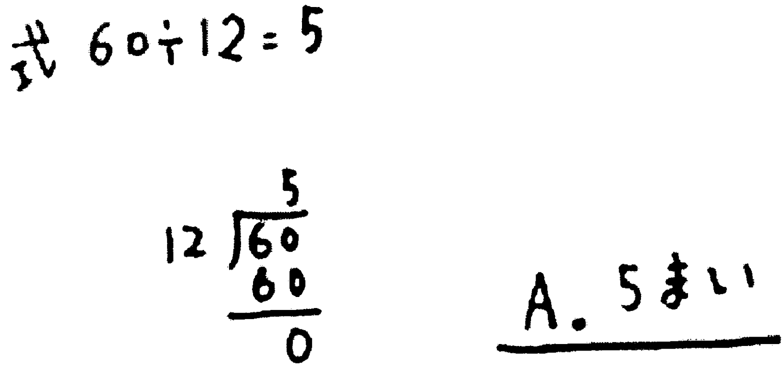

Figure 15.The top-left Chinese character means "number expression" and the bottom-right Japanese character means "5 papers".

3.1.2. Second Phase of Her Problem Solving (about 18 Minutes)

The interviewer decided to encourage her to make an image of this problem situation. He asked Maki whether she could imagine the situation where two children are dividing square papers. When Maki answered "Yes" to this question, the interviewer asked her to draw what she had in mind. Figure 16 is the drawing Maki drew in response to this request.

Maki explained this drawing as follows: the square at the left below represented the square papers the girl, Akiko, obtained, and the square in Akiko's hand represented the square papers Akiko was giving away to the boy, Kiyoshi. When the interviewer asked how many papers Akiko was giving to Kiyoshi, Maki muttered something but did not mention specific numbers of papers.When the interviewer asked whether they had done something before Akiko gave some papers to Kiyoshi, Maki answered immediately "As they divided, ...dividing." Then, the interviewer requested her to draw what she imagined (Figure 17).

Maki explained this drawing as follows: "[indicating the square between two persons] Take one paper from this...take one from this, put it on the papers Akiko will give Kiyoshi [the squares below the Akiko's legs]." When the interviewer asked what Akiko did first after taking a paper from that square, Maki said, "First, she put it on this [indicating the right square below Akiko], and then took one again and put it on this [indicating the left square below Akiko]. Akiko stopped it when no papers remained here [indicating the square between two persons]. When she finished this work, she took 12 papers from this [indicating the right square below Akiko] and put them on this [indicating the left square below Akiko]." When the interviewer asked how many papers each person took after dividing the 60 papers in this manner, Maki told that the half of 60 papers was 30 and wrote the following expressions and answer: "60–30=30, 30–12=18"; "60–30=30, 30+12=42"; "Kiyoshi can take 18 papers, and Akiko can take 42 papers." Seeing this answer written by Maki, the interviewer pointed to Figure 16 and told that according to her answer, the square in Akiko's hand, which represented the papers Akiko was giving Kiyoshi, became 18 papers and the square at left below, which represented the papers Akiko would take, became 42 papers. Maki seemed to accept this interviewer's comment.

Figure 16.

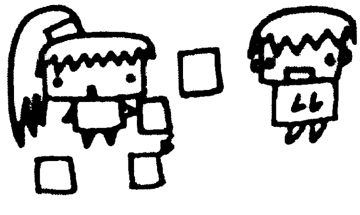

Figure 17.

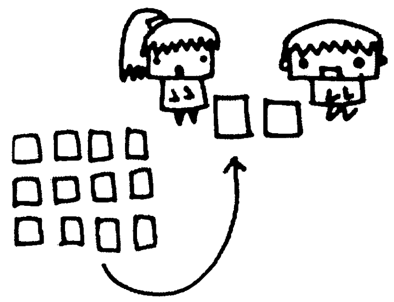

Figure 18.

3.1.3. Third Phase of Her Problem Solving (about 12 Minutes)

The interviewer asked what came into your head when she heard that "Akiko will get 12 more papers than Kiyoshi." Maki told him, "Kiyoshi and Akiko have the same number of papers, and then the number of Akiko's papers increases by 12 while the number of Kiyoshi's paper remains the same." The interviewer asked Maki to draw what she had just told him (Figure 18).

The interviewer encouraged Maki to use this drawing to find out the numbers of papers two persons would get. She wrote, "60÷2=30, 30+12=42" and "Akiko will get 42 papers and Kiyoshi keeps 30 papers." Then Maki and the interviewer talked about this solution as follows.

I: "Does your idea fit the condition of this problem?"

M: "Halving 60 papers... But, from 60 papers...As the Akiko's papers will be increased by 12, the total will become 72."

I: "So, if you have 72 papers..."

M: "I can divide the papers so that Akiko will get 12 more than Kiyoshi. But I have only 60, so if Akiko's papers increase by 12, Kiyoshi's papers decrease by 12."

3.1.4. Fourth Phase of Her Problem Solving (about 13 Minutes)

The interviewer pointed to Figure 18 and asked whether she could make a similar way of dividing within the given situation. Maki drew a largest square in Figure 19 and added a vertical line in it. Then she explained as follow:

M: "Six...I extract 3 papers from each person's papers, and I halve those 3 papers...not 3 papers...halve all of 6 papers like this. After that, this half piece can be considered one paper."

M: "Halving 3 papers out of Akiko's papers [and 3 papers out of Kiyoshi's], each person gets 6 papers and thus they'll get 12 papers altogether. Then, Akiko will take those 12 papers."

Maki also thought that the number of Kiyoshi's papers decreased to 27 because 3 of his papers had gone. When the interviewer asked how Akiko's papers changed, Maki extended Figure 19 to make Figure 20.

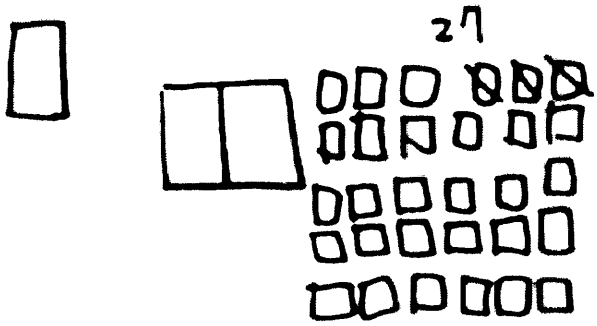

Figure 19.

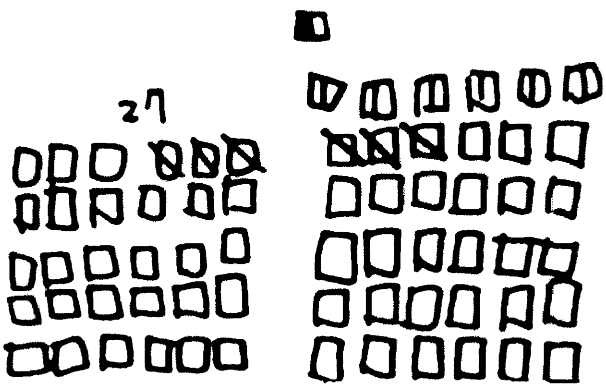

Figure 20.

Maki considered the numbers of Akiko's and Kiyoshi's papers to be 39 and 27 respectively based on this drawing. When the interviewer asked whether she had a feeling that the drawing in Figure 20 fitted the problem situation, Maki said after a while, "The problem does not say 'don't halve the papers'." She also said, however, "But I cannot make cranes if I halve those papers [because those papers are no longer squares]." A little later, Maki said, "Ah, I've got it," and newly drew 30 papers in Figure 21 and added slant lines to three of them. Then, she drew a left square and quartered it instead of halving it. She explained as follows: Thirty squares represented the Kiyoshi's papers; three papers were extracted and quartered to make 12 extra papers; those 12 papers would be given to Akiko.

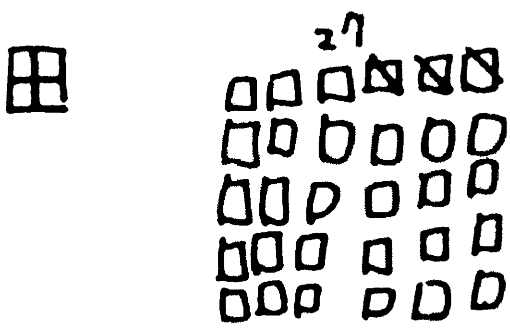

Figure 21.

3.1.5. Fifth Phase of Her Problem Solving (about 11and a Half Minutes)

Maki continued to think something, but she did not seem to complete her solution using the above idea. Thus, the interviewer asked whether she could divide the papers appropriately without cutting them. Maki answered, "Buying 12 more papers will make it possible."

The interviewer thought that it was necessary for her to experience the problem situation more concretely. He handed Maki 60 square papers and encouraged her to actually divide those papers.

Maki counted those papers to make a pile of 30 papers, and put the remaining 30 papers as the second pile. Maki said, "I need to make one of them 12 more, but I don't cut them, so I take 12 papers from this," and took 12 papers from the right pile and pretended to put those 12 papers in her hand on the left pile.

The interviewer directed her attention to Figure 18 and asked which square(s) of that drawing corresponded to the right pile, the left pile, and the 12 papers she had pretended to put on the left pile. Maki suddenly began to do the following actions: She gathered together the two piles except the above-mentioned 12 papers saying, "I'm stacking these papers"; She took one-by-one from the stacked papers and put them on a right and a left piles saying, "I'm doing it alternately"; After she made two new piles of papers, she finally put the taken-aside 12 papers on the left pile.

When the interviewer asked whether these piles of papers fitted the situation in Figure 18, Maki answered clearly"yes." When the interviewer asked the numbers of papers in the two piles, she answered 24 and 36 respectively.

Moreover, when the interviewer asked whether she could write numerical expressions for finding out those answers, Maki immediately wrote the following numerical expressions: "60-12=48, 48÷2=24, 24+12=36." In other words, Maki could solve the problem completely at this final phase.

3.2. Roles of the Student's Use of Drawings in the Interventional Problem Solving Session

3.2.1. Drawings as an Origin of the Main Idea

The changes in Maki's understanding and her use of drawings are summarized in Figure 22.

In the first phase of her problem solving, Maki calculated60÷12 and wrote only "5 papers" as the answer, although the problem asked the numbers of papers two children could take. Maki made the numerical expression here based on the keyword of the problem ("divide") and a plausible combination of the numbers given in the problem statement, not on the problem structure. Maki selected the divisor 12 because of the phrase "12 more" in the problem statement, and might not change the expression because the division 60÷12 produced a plausible answer, 5 papers.At the end of the session, Maki could make the appropriate numerical expressions which were the same as the expressions in Solution A mentioned above. She structured 60 papers so that the given conditions were satisfied and made the numerical expressions based on this structure. Maki's solution changed from superficial one to a solution based on the problem structure.

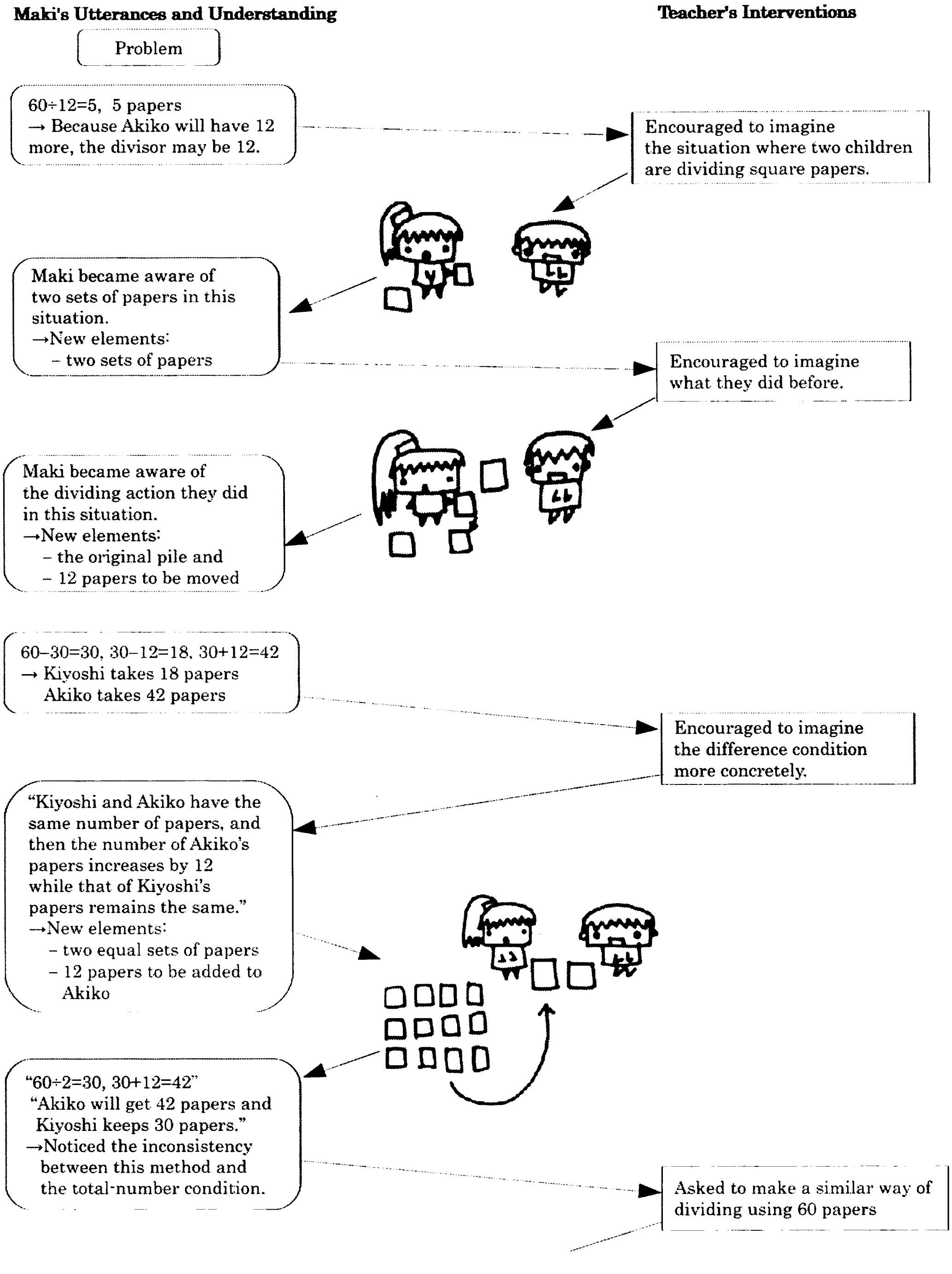

Figure 22. (Continued).

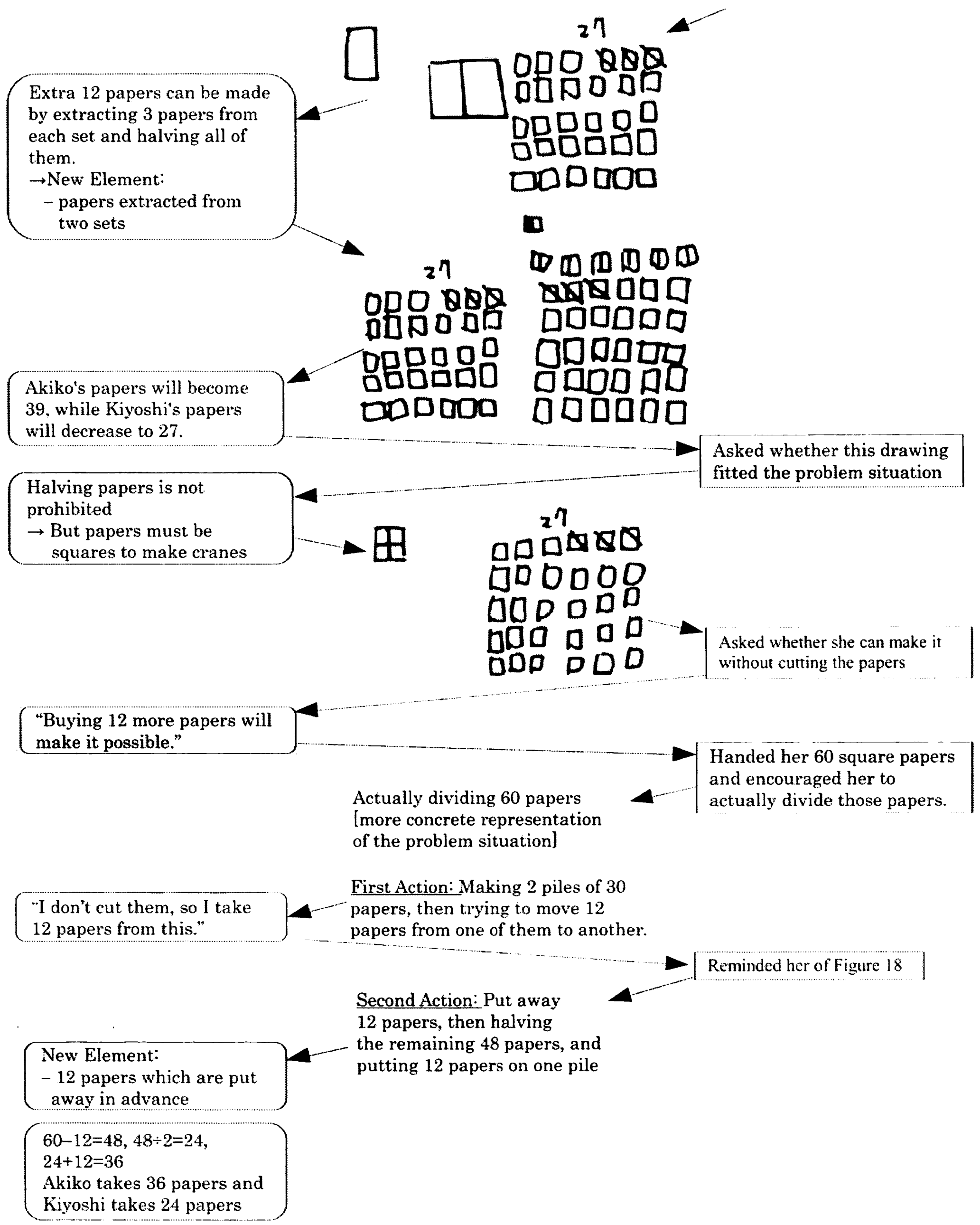

Figure 22.

She found this solution during dividing 60 papers. As Maki's first action of dividing 60 papers was based on the incorrect idea that 12 papers would be moved from one pile of 30 papers to another pile, it was not so helpful for her problem solving. Only when her action was organized by Figure 18, Maki's action of dividing papers became helpful for her problem solving. She might notice that because the 12 papers Maki had in her hand at that time corresponded to the 12 papers beside Akiko in Figure 18, two piles which consisted of the remaining papers had to be of the same size as Figure 18 illustrated. Therefore, Maki grouped together the two piles of papers, which had 18 and 30 papers respectively, and newly divided those 48 papers into two piles of 24 papers. In her second action, Maki attempted to make the critical elements of the problem situation, two equally-sized piles of papers and 12 extra papers, which were drawn in Figure 18 in order to represent the difference condition. In other words, the drawing in Figure 18 introduced the difference condition to the 60 papers Maki was dividing. Since they had naturally embodied the total-number condition, those 60 papers became the fully structured representation of the problem situation when they embodied Figure 18. In the sense that it gave the necessary structure to Maki's dividing action and her second action was a realization of Figure 18 using 60 papers, the drawing in Figure 18 played a significant role in finding the main idea of her solution, dividing the papers after putting 12 papers aside, even though it was more pictorial than the standard diagram shown in Figure 14.

Even though the drawings Maki drew at the later stages of her problem solving process did not seem so helpful for her solution, some important ideas emerged in those drawings. When drawing Figure 19 and extending it to Figure 20, Maki attempted to make extra 12 papers from two equally-sized piles of 30 papers. Maki realized that she could make such 12 papers by extracting 3 papers from each pile (i.e. 6 papers altogether) and halving all of them. If she had noticed that she could make 12 papers without cutting papers by extracting 6 papers from each pile, Maki could have solved this problem in the manner similar to Solution B. Because her use of drawings made it possible for her to find such an idea which could have led to another solution, it can be said that her use of drawings could play an important role in her problem solving. It should be also noted here that in her attempt to make 12 papers from equally-sized piles of papers, Maki extracted the same number of papers from both piles and kept those piles equal. As discussed above, this basic structure consisting oftwo equally-sized piles of papers and 12 extra papers was critical for Solution A and used for organizing her dividing action. Figure 20 might help her be aware of this basic structure by showing two piles in a symmetrical way.

3.2.2. The Intervention Based on the Changes in Drawings

In this interventional session, the teacher's interventions were based on the framework presented in Section 1. He assumed that the student's drawings could be elaborated gradually and her understanding of the problem situation could also be elaborated in the process of constructing and modifying those drawings.

Because, as discussed in the following, Maki could deepen her understanding of the problem situation with such teacher's interventions and solve the problem based on her full-fledged understanding, it can be said that the teacher's interventions were successful to some extent.

It was observed that her drawings changed gradually and became to represent the problem situation more appropriately. The drawing Maki drew at the later stage, Figure 20, included only the square papers and excluded the pictures of the children.

It represented a numerical relationship between the total number of papers, two equally-divided sets of papers and the 12 papers to be moved later to Akiko, even though the represented relationship was incomplete and represented in a little pictorial manner. That is, the drawing in Figure 20 can be considered to be more schematic than the drawings Maki made at the earlier stages (i.e. Figure 16, 17 and 18). The above-mentioned elements of Figure 20 had emerged in Figure 18 without the information about the numbers of equally-divided sets. Maki drew Figure 20 in order to make a similar situation using 60 papers. Figure 20 can be seen a kind of modification of Figure 18 for incorporating the total-number condition into Figure 18.

Although this drawing, Figure 18, had a pictorial flavor, it also represented the critical elements in the problem situation, as discussed above: two equal sets of papers for Akiko and for Kiyoshi and the 12 papers to be added later. These elements correspond to the three line segments of the line diagram in Figure 14. Thus, Figure 18 can be considered more schematic than her first and second drawings, Figure 16 and 17. Figure 18 can be also considered a kind of modification of Figure 17, because these two equal-size sets emerged in drawing Figure 17.

According to the Maki's explanation about Figure 17, Akiko took papers one-by-one, put them on two piles of papers alternately, and stopped dividing when no papers remained. Maki thought that two squares under Akiko in Figure 17 became two equal-size sets of 30 papers when Akiko finished her action of dividing 60 papers. Two equal-size sets of papers in Figure 18 can be considered an expression of the equal-size sets which emerged as a consequence of Akiko's dividing action. In her attempt to make explicit the difference condition in a drawing, Maki modified the previous drawing into Figure 18. Similarly, it can be said that Figure 20 represented the story which happened after Figure 17.

Maki's first and second drawings seemed pictures of the problem situation. But these drawings also included important elements in the problem situation. The drawing in Figure 17 included the pile of papers from which Akiko took papers one-by-one and the Akiko's action of dividing those papers. This pile of papers was used in Maki's action of dividing actual 60 papers at the last stage of her problem solving, and her awareness of this Akiko's action directed her attention to two equal-size sets as discussed above.

Figure 16 included two sets of papers which Akiko and Kiyoshi would take respectively. Recalling that Maki wrote only one number, "5 papers," as the answer in her initial solution, this drawing played an important role in her problem solving in that it could remind her of the existence of two sets of papers, Akiko's papers and Kiyoshi's papers, in this problem situation.

It might take less time for Maki to arrive at her final solution if the teachers intervened more directly, for example, by presenting a template of the line diagram and helping her complete that diagram. Although they were less efficient in this sense, the teacher's interventions which encouraged her to imagine the problem situation and express those images could trigger her exploration of the problem situation and help her understand it more deeply.

If the actual operation on 60 papers can be seen a kind of external representation or a simulation of the problem situation (*cf.* Nunokawa, 1997), it can be concluded that the interaction of Maki's use of representations and her understanding of the problem situation was observed throughout Maki's problem solving when the teacher intervened by encouraging her to proceed through this interaction.

3.2.3. Limitations of Maki's Use of Drawings and Their Origin

Some differences between Maki's and the pair's uses of drawings were observed. First, while the students in the pair began to use drawings spontaneously, Maki began to use drawings only after she was prompted by the teacher. Second, while the students in the pair constantly cared about the reasonableness of the results of their calculations using their drawings, Maki did not mention such reasonableness until the teacher asked it. These differences are closely related to their metacognitive activities, and the differences among their metacognitive activities must have partly been driven by the difference between the problem solving settings which they participated in: the collaborative problem solving of the pair students could facilitate their metacognitive activities in their interaction, while Maki who solved the problem individually did not have a partner monitoring and controlling her thinking.

But it is possible to consider these differences to be a consequence of a different factor: Being oriented to understanding of problem situations. If students are willing to understand problem situations, those students are more likely to construct or change drawings to explore those problem situations when they are stuck in their problem solving. They are more likely to examine the reasonableness of their results in the problem situations when they find out some results.

For example, when the pair of students performed the calculations derived from the initial interpretation of Figure 6, they examined the result of those calculations. As they found that the result did not fit the problem situation, Shiho said, "I'd better go back to the beginning and start over," and began to explore the problem situation again. On the other hand, when she obtained a result based on her initial understanding of Figure 18, Maki noticed that 72 papers were necessary for her method only after the teacher directed her attention to the reasonableness of her solution. Even in that case, she did not explore the problem situation per se, and instead, she introduced new conditions, cutting papers and buying 12 papers, which would change the problem situation totally.

This factor can also explain why Maki needed actual papers to finish her solution. Because Maki was less oriented to understanding of the problem situation than Shiho, she needed the representation which could more directly lead her to exploration of the problem situation.

Because actual 60 papers were easier to operate than drawings and made it possible for her to simulate the problem situation, Maki could be aware of the problem situation more clearly and find how to realize the situation of Figure 18 using only 60 papers.

As the interventions the teacher adopted in this interventional session was consistent with the standpoint that understanding of problem situations is most important in solving mathematical problems (Nunokawa, 2001, 2005), those interventions could compensate Maki's lack of willingness to understand the problem situation.

CONCLUSION

The analysis of the problem solving process of the pair of students showed that a kind of interactive process, in which the students' understanding of the problem situation and theiruses of drawings changed each other, was observed even in the problem solving of elementary school students and that the basic framework about use of drawings presented in Section 1 can be applied to problem solving at the elementary school level. The analysisof the interventional session showed that the teacher's interventions based on this framework was effective, to some extent, for supporting an elementary school student's problem solving. Those interventions could help a 5th grade student solve a difficult mathematical problem not by providing her with a hint or a schematic diagram which could lead her to a solution in a direct manner. but by encouraging her to proceed through the above interactive process.

The problem solving processes analyzed in this chapter can be considered the examples which embodied "an interactionof internalization of external representations and externalization of mental images" (PapeandTchoshanov, 2001) in the context of mathematical problem solving at the elementary school level. But what is important here is the fact that the students' understanding of the problem situations and their uses of drawings deepened each other and, in those interactive processes, pictorial drawings or insufficient drawings played certain roles to drive those interactions. Even though those insufficient drawings could not lead students to their final solutions immediately, they could suggest some pieces of information about elements or relations in the problem situations, make explicit inconsistencies of their partial findings and the problem situations, and lead students to revising their understandings. Through such interactive processes, the students' drawings changed gradually into ones which reflected the problem structures and implied the critical ideas for their final solutions.

While the difference between the students' uses of drawings in the collaborative and the individual problem solving could be explained by the difference of the easiness of metacognitive activities in those settings, it could also be explained by the difference of the students' willingness to understand the problem situations. Taking account of the results of the above analyses and adopting a standpoint that understanding of problem situations should be central in mathematical problem solving, what is most important in solvers' uses of drawings is not whether their drawings are pictorial or schematic, but whether solves use their drawings with willingness to understand problem situations better.

In the long term, students are expected to use drawingsflexibly in order to facilitate their own problem solving. Providing students with opportunities to experience the above-mentioned interactive processes and appropriate those processes can be a way of teaching students to use drawings, which is an alternative to teaching the appropriate use of schematic diagrams. According to the two explanations about the difference presented in Section 4.2.3, two dispositions should be developed in students in such a teaching. Concerning metacognitive aspects, students need to learn to manage their uses of drawings. They will be required to begin their uses of drawings spontaneously, direct their attention to what drawings imply, and, if necessary, lead themselves to reexamination or reconstruction of drawings. Concerning the willingness to understand problem situations, students need to have an appropriate belief about mathematical problem solving: Their understanding of problem situations is critical to mathematical problem solving. It is necessary for teachers to provide their students with problem solving experiences in which they can construct such a belief and become more willing to explore and understand problem situations. These two dispositions are dependent on each other (Schoenfeld, 1992): Students'management of their uses of drawings must be influenced by their beliefs about mathematical problem solving, while what kinds of management are highly evaluated shape students' beliefs. At least in the context of mathematical problem solving, students' uses of drawings is not the end of problem solving, but the mere means to it. Means should be organized to proceed to the end. If "close attention to what students say and do in relation to what a teacher says helps usunderstand the details of practice that matter for student learning" (Webb *et al.*, 2009, p. 66), it can be said that the analyses conducted in this chapter will help us understand the details of practice which matter for students' uses of drawings as a means to mathematical problem solving.

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