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Empirical and Autonomical Aspects of School Mathematics

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The aim of this paper is to re-examine the importance of autonomical aspects of school mathematics. First, two aspects of mathematics, empirical and autonomical, will be found in some philosophical researches. Then, some discussions in mathematics education will be organized based on these two aspects, and examples from ordinary school mathematics contents will be re-examined to show the importance of autonomical aspects even in school mathematics. Finally, compatibility of this paper with recent tones in mathematics education will be checked.

Recently, it may be required that school education should be related to everyday or real situations more closely, as reflected in integrated or cross curricular, and such tones seem to influence mathematics education. For example, some tendencies showing closer relations between mathematics and real situations can be seen in the review of recent researches in mathematics education (De Corte et al., 1996). This paper will re-examine a relation between mathematics and real situations to discuss these issues soundly, and will try to show that it is theoretical or autonomical aspects of mathematics to make mathematics useful in real situations.

EMPIRICAL AND AUTONOMICAL ASPECTS OF MATHEMATICS

It is generally said that mathematics has two aspects; (i) description of real world or the solution of practical problems; (ii) symbolically represented structures independent of their real-world roots (DeCorte et al., 1996, p. 500). According to Høyrup (1994), "[i]t must be the duty of the teacher, from primary school to university, to present a double picture of mathematics: as a field of human knowledge with its own *integrity*, requiring its own autonomous further development; but a field which at the same time must achieve *integration* as part of human knowledge and human life in general" (p.22, original italics).

In explaining the fact that mathematics is applicable to questions of the real world, Rav (1993) says that the core logico-operational component of mathematics has its empirical basis, which have evolved by confrontation with and adaptation to the world and have become fixed in the course of evolution, and that specific contents of mathematics have practical origins, including cultural needs (p. 89). Maddy (1991) takes as one of the items of agreement in recent philosophy of mathematics the point that a very basic level of epistemic contact with modified Platonistic ontology is available through familiar channels like ordinary perception (p. 157). Even Goodstein (1970), who criticizes the empiricist view of the foundation of mathematics, accepts the empirical basis for parts of mathematics; "Mathematics is applicable for the very reasons that empiricists hold mathematics to be empirical, namely that some parts of mathematics are abstractions from experience" (p. 55).

Some researchers mentions the importance of formal mathematical method in discoveries of physics (e.g. Steiner, 1989). Zahar (1980) points out that intuitive physical principles can acquire some surplus structure when they are translated into mathematical theory. Moreover, he presents the importance of autonomical aspect of mathematics in discovering a physical theory; "The physicist operates at the mathematical level, hoping that his operations mirror certain features of reality. However, he is not very as to how this mirroring take place, so he lets himself be guided by the syntax, or by the symbolism, of some mathematical system (pp. 6-7). Rav (1993) also mentions a similar point; "As the pattern and structures that mathematics consists of are molded by the logico-operational neural mechanisms, these abstract patterns and structures acquire the status of potential cognitive schemes for forming abstract hypothetical world pictures. Mathematics is a singularly rich cognition pool of humankind from which schemes can be drawn for formulating theories that deal with phenomena that lie outside of daily experiences and, hence, for which ordinary language is inadequate" (p. 92). Mathematics is "patterns we human beings develop to help us comprehend the world" (Devlin, 1994, p.74), and the autonomical aspect of mathematics seems to allow us to go beyond our everyday observations.

According to Lakatos (1976), mathematics has its autonomy in nature; "Mathematics, this product of human activity, 'alienates itself' from the human activity which has been producing it. It becomes a living, growing organism, that *acquires a certain autonomy* from the activity which has produced it; it develops its own autonomous laws of growth, its own dialectic” (p.146; original italics). As Restivo (1993) points out, it is needed to take account of "the importance of interplay between the centripetal forces of closure and autonomy and the centrifugal forces that maintain connections to the widest range of worlds" (p.266). It implies that autonomical aspect of mathematics is also important and it is this aspect that supports many applications of mathematics to the real world or other deciplines. According to Høyrup (1994), in order to solve immense problems the contemporary world is confronted with, "mathematics must be allowed to develop, and mathematical development furthered at our best; according to historical experience mathematics must hence be allowed autonomous existence" (p.22).

TWO ASPECTS IN SCHOOL MATHEMATICS

In the previous section, two aspects of mathematics, empirical and autonomical aspects, was found in the literature about philosophy of mathematics. In this section, those aspects will be sought in mathematics education, especially focusing on mathematical problem solving.

Some researchers recommend that learning mathematics and understanding mathematical knowledge can be supported or facilitated by real situations. For example, Resnick (1988) intends to "use children's knowledge of the relationship between stories and expressions to help them understand the reasons for symbolic algebra rules such as the 'sign change rule'" (p. 37). She mentions the instruction focusing on the task of interpreting mathematical expressions as mathematizations of possible real-world situations. Schroeder & Lester (1989) shows a modification of the problem-solving model for translating problems. In this model, the mathematical processes are considered to be "under construction," and its most important features are the relationships between the steps in the mathematical process and the actions on particular elements in the problems (p. 36). Such a view can be represented by the following diagram.

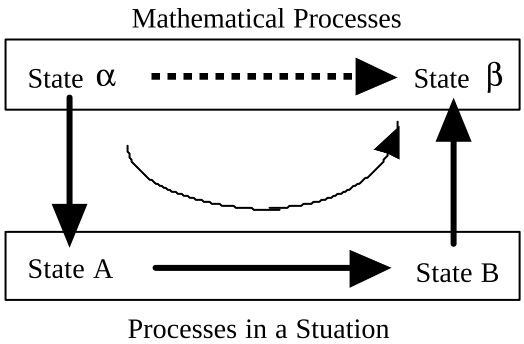

Fig.1

In this diagram, mathematical processes from State α to State β are supported or determined by referring to the processes in a situation from State A to State B. Activities involving materials for understanding (e.g. blocks or tiles for understanding algorithms of addition and subtraction) can be also seen to lie in this pattern.

Street mathematics or informal knowledge (e.g. Nunes, 1992) is related to this pattern. People can implement, in a real situation, some processes corresponding to a certain kind of mathematical processes. The researcher can find this correspondence in the people's processing, but the people themselves are not aware of this (see Fig. 2).

The discussions on 'theorem in action' (Vergnaud, 1982) also suggest that some of mathematical ideas or theorems are used in students' empirical reasoning, although they are not aware of them, and that those ideas and theorems may be derived from students' experiences.

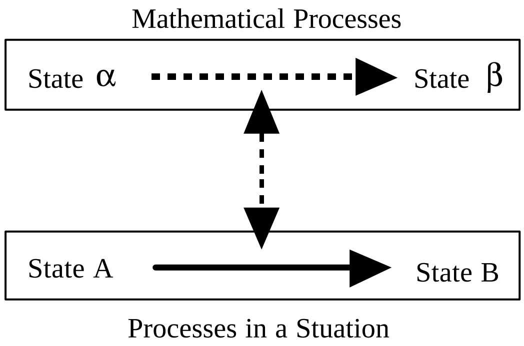

Fig. 2

In the mathematical problem solving, especially one focusing on application of mathematics to real world or mathematical modeling (cf. Nunokawa, 1995; Silver, 1992), the relationship between processes in mathematics and in situations can be represented like following, which should be clearly distinguished from Fig. 1.

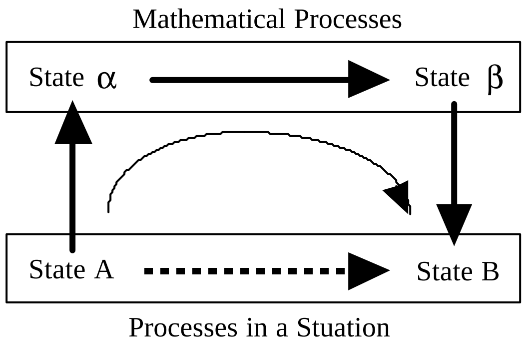

Fig. 3

If we recall here that mathematics should be used not to find a final solution but to give information for discussing the situation better and making more sensible decisions (Fischer, 1993), the mathematical processes form State α to State β must generate some pieces of information which the processes in a situation cannot generate or which are difficult for those processes to generate. Such information produced through mathematical processes may allow students to feel the usefulness or values of mathematics in real situations. So, in order to emphasize this point, the above diagram can be modified like Fig. 4.

In this diagram, the new state in the situation (State C) is generated through mathematical processes and processes from State B to State C is absent in the situation. The diagram implies that mathematical processes which does not necessarily correspond to situational processes are needed so that they can produce really new information. This is consistent with Høyrup's (1994) comment cited above, and the necessity of autonomical aspects of mathematics is rising to the surface.

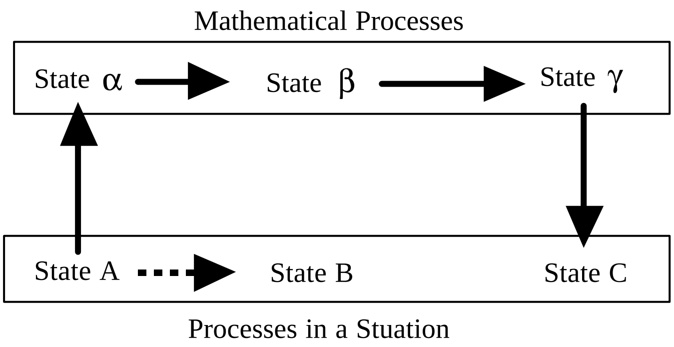

Fig. 4

Although school mathematics can be built and learned by referring to or being supported by real situations, like Fig. 1 or Fig. 2, it needs to be developed autonomically so that it can have mathematical knowledge or processes which are not supported by real situations and can produce new information about situations, like Fig. 4. We should note here that autonomical aspects of school mathematics play an important role in effectively connecting mathematics with real situations.

ROLES OF AUTONOMICAL ASPECTS IN SCHOOL MATHEMATICS CONTENTS

The phenomena where new information is produced through autonomical aspect of mathematics can be found in ordinary school mathematics contents, outside problem solving context. We sill see some examples in this section.

Example 1. Computation of Integers

Basic facts, e.g. 3+5=8, can be induced from real situations by counting the number of objects in the combined set consisting of three-object set and five-object set. The rule of carrying can be basically determined based on the decimal notation of numbers. This rule must be consistent with the basic facts induced from real situations, but it can be defined on the way how numbers are described in our notation. Some researchers recommend the use of blocks or tiles (e.g. Hiebert & Carpenter (1992)), which are real objects in a sense. We should note here, however, that these materials and the operations on them are one kind of symbol system and they cannot determine the numerical notation and operations on it. Moreover, those materials cannot make sense to the students unless the students establish the isomorphism between the materials and numerical notation. It is the way of description in the numerical notation that determines the way of carrying.

Then the way of computation for larger numbers can be constructed using elementary basic facts, which are now among mathematical knowledge, and the way of carrying. In other words, this way of computation for larger numbers can grow autonomically once the basic facts and the way of carrying are determined.

Using these basic facts and the rule of carrying, we can compute the addition of larger numbers, e.g. 1,562,356,692,738+328,991,225,623. If we use this computation to find the number of the marbles in the combined set consisting of the sets having these numbers of marbles, its result might be seen as a kind of prospect, since we need to count the objects in the combined set in order to know the exact number of those objects. But we can get very good prospects by implementing the computation and they are reliable, then they are usually treated as real numbers of objects in the combined sets. And more places or digits the rule is formulated to, more situations we can deal with. This suggests that autonomical aspect of mathematics supports application of mathematics to real situation.

Resnik (1990) writes that "in computation I see the closest connection between our purely mathematical beliefs and mathematical objects" (p. 63). If we take the result of addition of two large numbers to be a mathematical object and 'knowing which number the result is' to be a our mathematical belief, Resnik's (1990) argument suggests that our computation itself is a reason we come to believe that one number is the result of that addition, which may make a certain number the result of that addition. So, the computation process is critical to getting the result of computation, although it can be practically omitted by using calculators.

The importance of the autonomical aspect in the application of mathematics is clearer when dividing computation is mentioned. The algorithm of the division can be constructed based on the basic facts and the rule of carrying or borrowing. The latter is determined by our numerical notations, as mentioned above. The way of moving or deleting the decimal points can be also established based on the notation, especially the unit of each place. When we apply the division to a real situation, we can make a prospect about the situation which is based on the more drastic change of the situation than the change which we can get when using the subtraction (see Nunokawa (1995), p. 726). For instance, consider the situation where 3,682 marbles are being divided so that each child has 7 marbles. This situation can be treated using subtraction if it is repeated. When the division is applied to the same situation, however, only one step is needed, and the situation, where there is one pile of 3,682 marbles, can be changed into new one, where there are 526 piles of 7 marbles. This means that, if we develop our world of computation to include the division within it, we can have a chance to make more interesting prospect and use a bigger power of mathematics. Since the division algorithm is constructed based on the autonomical aspect of mathematics, this example shows that developing mathematics autonomically may support our application of mathematics to situations.

According to Vygotsky (1978), such a process can be characterized as the reverse of the relation between action and meaning (p. 100). That is, whereas, in the earlier stage, the action embedded in a situation dominates, afterwards the meaning of the action become detached from action and dominant. In this stage, people may act following the meaning, and "begin to act independently of what he sees" (p. 97).This emancipation from situated action or appearance would make it possible for people to apply the meaning or rules to broader areas beyond what they see.

Example 2. Areas of Geometrical Figures

In elementary mathematics, the concept of area is developed based on the everyday concept of extent and formulated as number of unit squares. This formulation can lead to establishing the formula of rectangular area, which in turn derives the formulae of areas of other geometric figures. Such development of the formulae is considered autonomical aspect of school mathematics. In establishing the formulae of squares' or rectangles' areas, indeed, our everyday knowledge or intuition about extent of things may be used. Once they are established, the formulae for other kinds of figures can be induced using those basic formulae and other mathematical knowledge, e.g. knowledge about equivalent transformation. That is, they are usually established and validated within students' mathematics system. For instance, in Japanese textbooks, the formula of circles' areas is obtained based on the formula of parallelograms' areas which can be deduced from rectangles' formula.

When the formulae are used in finding the area of land or something, this use of the formulae can be considered a kind of mathematical modeling in the sense that which formula be used and how it be used depend on what kind of figures the user sees in those things. The result of application of a formula and its computation can be taken as prospect about the extent of the land or something.

More kinds of figures we can have the formulae of areas for, more possibility we can have to apply our mathematics to real situation and more accurate our prospect about the objects' extent can become. Furthermore, it may be possible to compare the extents of things among which we cannot choose at glance the larger one, e.g. a circular thing whose radius is a half meter long and a rectangular thing whose width and length are 1.7 and 1.8 meter respectively. This means that autonomical aspect of mathematics may provide us with new information about real situations which may be difficult to be reached without mathematics, and support our mathematical power. As Bodrova & Leong (1996) put out, "the everyday concept is changed by the learning of the scientific concept" (p. 60).

Example 3. Deductive System of Geometry

Some experts have rich bodies of knowledge about shapes; for instance, if two diagonals of the lid of a box have the same length, then that lid is neat one; and the places where those diagonals cut each other is the center of the lid (Millroy, 1992, pp. 98-100). By the way, the concepts of geometric figures can be considered scientific concepts developed based on children's everyday concepts and formulated in a form of definitions (cf. Nunokawa, 1993). In geometry, figures' properties are deduced and proved by using only their definitions. Knuth (1985) thinks that constructing chains of implications is analogous to constructing chains of computer instructions that transform some input into some desired output, using a repertoire of subroutines (p.174). This implies that making proofs can be seen as autonomical growth of mathematical knowledge. In this system, the fact that, if two diagonals of a quadrangle have the same length and they cut each other into halves, then the quadrangle is a rectangle can be proved using only the definitions or facts within that system. If one models the lid of a box as a geometric quadrangle, then he can find out whether it is rectangular or not by examining its two diagonals. In other words, autonomical aspect of school mathematics allows one to use the way of decision making which is similar to experts' wisdom. Further this system is developed, more 'wisdom' we can use in dealing various shapes.

Since many of our everyday items seems to be designed using geometric figures, especially rectangles or circles, rich pool of autonomical aspect of mathematics might make it possible to produce rich pieces of new information when those items are modeled as geometric figures.

Example 4. Functional Reasoning

In the discussions about mathematical modeling in mathematics education, emphasis seems to be placed on the reality-mathematics interfaces, i.e. translation of reality into mathematics and interpretation of mathematical results (e.g. see Nunokawa, 1995). We should also note, however, that it is the autonomical aspect of mathematics to make such mathematical modeling processes useful and powerful. Let us consider as a example the baton-pass problem of Yanagimoto et al. (1993). They used this problem in the 9th grade math classes and asked students to decide the best position of the markpoint for the succeeding runner of relay based on the data the student had collected (Succeeding runner will begin to run when the preceding runner reach this markpoint). In this modeling process, the motion of the preceding runner and succeeding runner was modeled by the linear and quadratic function respectively. As the condition on which two curves contact each other gives the best markpoint, this condition must be found out by calculating the discriminant of the quadratic equation, for example.

Therefore, a certain theory of linear and quadratic function, especially of their intersection and contact, is needed for mathematics to produce useful information in this realistic problem situation. Such mathematical knowledge may be developed autonomically in students' mathematics system.

As mentioned above, producing new information about obtained functions usually requires solving equations, which, in turn, requires symbolic manipulations (cf. Miwa, 1996). Even if such manipulations are introduced empirically (e.g. a "balance" metaphor), they are established as an autonomical aspect. In other words, this establishment is required to get new information about functions[1]. Further we can transform expressions, more information we can get about functions and real situations.

Yanagimoto et al. (1993) reports that students showed the strongest enthusiasm and interest in the baton-pass problem where more advanced mathematical knowledge was used than other realistic problem situations, "presumably because the problem was to improve their own relay performance" (p. 124). This implies that important is not the close connection between reality and mathematics but the usefulness of mathematics for improving students' performance. And this usefulness was supported by new information produced through an autonomical aspect of mathematics.

COMPATIBILITY WITH RECENT TONES

At a first glance, the basic claim of this paper, emphasizing the importance of autonomical aspects of school mathematics, seems to go against the recent movements in mathematics education (e.g. decreasing emphasis on computation). Finally, we would check compatibility of this claim with some of recent tones in mathematics education.

Hiebert et al. (1996) emphasize that "mathematical content be considered seriously when selecting tasks and that the definition of usefulness be expanded to a variety of problem situations, including those contextualized entirely within mathematics" (p. 18). Autonomical development of school mathematics can be seen as tasks contextualized entirely within mathematics, so it can be selected from their perspective. In that context, our attention may be paid to that selected tasks can lead to interesting mathematical facts, which implies further autonomical development of students' mathematics.

Some researchers recommend that mathematics can be considered science of pattern and school mathematics should be designed based on this view (e.g. MSEB, 1990). It should be noted here that the task of science of pattern may not only be finding some patterns in various situations, but also be developing "theories" about those patterns, e.g. exploring their characteristic, studying their relation to other patterns, extending them to construct new patterns. The development of autonomical aspect of mathematics results in making richer the bodies of knowledge about some mathematical patterns, so it implies the development of students' 'theories' about them.

Brown et al. (1989), who are prominent proponents of situated cognition, locate "generality" on the sequence of students' progress, and say that future work into situated cognition must try to frame between explicit and conceptual knowledge and implicit activity-based understanding. They also mention the similarity between concepts and tools, and say that both of them "can only be fully understood through use" (p. 33). If we take development of autonomical aspects as brushing up or improving tools, emphasis of that development seems compatible with their claims, as far as using mathematical tools in real situations is taken account of (This 'tool' metaphor is also compatible with scheme shown in Fig. 4).

Following such discussions, developing students' mathematics autonomically can be compatible with recent tones of mathematics education. What is important concerning this issue is rethinking roles of autonomical aspects, and locating those aspects properly in school mathematics.

CONCLUDING REMARKS

Masingila et al. (1996) suggests that , to help students acquire the concepts and skills that are useful to solve dilemmas people encounter in life, generalization of students' out-of-school mathematics is needed (p. 194). The above consideration implies that developing students' mathematics autonomically, going beyond simple generalization of it, is important if students' mathematics is to be seen as a powerful tool for dealing with real world. We should respect experts' intuition or insight, and they might not need mathematical tools in dealing their familiar situations. Such mathematical tools, however, allow novices to make decisions rather precisely as experts do. It means that school mathematics can extend persons' competence, and this extension is supported by the autonomical aspect.

Autonomical aspects are not directly related to children's real world, but it can mediate between children and real world and change the relationships between them (see Brown et al., 1989, p. 33). To achieve this mediation, when we pay attention to some autonomical aspects of mathematics, at the same time, we should take account of their roles in real situations which give us "flexibility and power in dealing with particulars" (Egan, 1997, p. 236)[2].

NOTES

1. Although graphic calculator can provide us with alternative procedures (e.g. operations on graphs or numerical approximations), those procedures are some kinds of operations on

mathematical objects (graphs or expressions). They are also based on key-operations of calculator instead of symbolic manipulations. Symbolic manipulation software can implement those manipulations automatically. But, whether or not students themselves implement them, those manipulations needs to be implemented. What such technologies suggest is not that autonomical manipulations are unnecessary, but that we should pay attention to what kind of manipulations we would prepare for our students and whom we would assume to implement them.

2. When students need not to implement manipulations, as in the context of computer using, quality of produced new information may come into question more clearly.

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