MICROANALYSIS OF THE WAYS OF USING SIMPLER PROBLEMS IN MATHEMATICAL PROBLEM SOLVING[1]

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The aim of this paper is to examine how solvers use solutions of simpler problems to explore original problems. According to analysis of data by two problem solvers, it will be stated that; (i) solutions of simpler problems can suggest some aspects to which solvers should pay attention in exploring an original problem situation, and can support importance of some elements in the situation; (ii) it is an important factor in using solutions of simpler problems to explore original situations and get information about it ; (iii) although solvers' making-sense of solutions of simpler problems plays a crucial role, inappropriate making-sense does not need to lead to failure and can promote solvers' activities.

1. Introduction

It is widely recognized as a strategy to use similar and simpler problems in solving mathematical problems. Yokoyama (1991) found, however, that teaching of this strategy did not have so much effect on children as teaching of other strategies such as guess-and-check, making-lists, and working-backward. While Schoenfeld (1985) showed that students could use solutions of simpler problems effectively in solving original ones, Tsukahara (1991) reported students' difficulty in using those solutions.

Such research did not focus on processes themselves in which students investigate original problems taking advantage of solutions of their simpler versions. This paper will attempt to examine these processes as such by analyzing the protocols of the actual mathematical problem solving, to understand roles of simpler problems better.

2. Gathering Data and the Outlines of the Solutions

2.1 Gathering Data

The problem solving processes of two solvers (call them the subjects S and T in the rest of this paper) will be treated in this paper. Each of these solvers participated in the problem solving experiments consisting of nine sessions. Both of them were graduate school students studying mathematics education, and S is the same person as the subject in Nunokawa (1994b). What will be analyzed here are the data of the third session of each experiment. In this session, the following problem was tackled;

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Prove that if *a*, *b*, and *c* are positive real numbers, then

$$a^a b^b c^c \geq (abc)^{\frac{a+b+c}{3}}$$

(Klamkin, 1988, p. 5).

The subjects were asked to solve this in the think-aloud fashion. The whole solving processes and interviews were recorded by ATR and VTR. The transcriptions of these records, answer sheets written by the solvers, and memos taken by the experimenter are used for analysis (for detail, see Nunokawa (1994b)).

2. 2 The Outline of the S's Solving Process

(i) He transformed the right-side of the inequality to be proved, and got $a^{\frac{a}{3}}a^{\frac{b}{3}}a^{\frac{c}{3}}b^{\frac{a}{3}}b^{\frac{b}{3}}b^{\frac{c}{3}}c^{\frac{a}{3}}c^{\frac{b}{3}}c^{\frac{c}{3}}$. Then he took the logarithms of the both-sides and multiplied them by 3 to get $3a\log a+3b\log b+3c\log c$ and $(a+b+c)\log a+(a+b+c)\log b+(a+b+c)\log c$, respectively. He introduced the condition $a>b>c>0$ by himself, saying "since it does not lose generality." Here, he mentioned $3a-(a+b+c)>0$ and $3c-(a+b+c)<0$, but said that he could not decide whether the rest (i.e.$3b-(a+b+c)$) was positive or negative. Saying "I would try to subtract [($a+b+c$) from $3a$, and so on]," he wrote $2a-(b+c)$, $2b-(a+c)$, and $2c-(a+b)$, and added the mark "O"(indicating 'OK') to the first one, "×"('not OK') to the second and third ones.

(ii) He examined $0.5^{0.5}$ saying "Does the greater-less relation reverses at 1?" Then he drew a graph of $y=x^x$. Based on the fact that the minimum of $y=x^x$ was $(e^{-1})^{e^{-1}}$, he estimated the left-side as $a^a b^b c^c \geq (e^{-1}e^{-1}e^{-1})^{e^{-1}}=e^{-3e^{-1}}$, saying "it cannot be less than this." After that, he began to search for an alternative approach.

(iii) Saying "I'd try another idea," he examined the one-letter and two-letter cases of the given inequality. He took the logarithms of the both sides of $a^a b^b \geq (ab)^{\frac{a+b}{2}}$, and subtracted its right-side from the left-side. After transforming it into $(a-b)\log a+(b-a)\log b$, and into $(a-b)(\log a-\log b)$, he noted that the two-letter case had been proved because $(a-b)$ and $(\log a-\log b)$ had the same sign.

(iv) Adopting "the same policy," he wrote $3a\log a+3b\log b+3c\log c-(a+b+c)\{\log a+\log b+\log c\}$ and transformed this into $(2a-(b+c))\log a+(2b-(a+c))\log b+(2c-(a+b))\log c$. He transformed $2a-(b+c)$, $2b-(a+c)$, and $2c-(a+b)$ into $a+a-b-c$, $b-a+b-c$, and $c-a-b+c$ respectively, and examined them. During this examination, he marked the sign "+" on the first expression, "?" on the second, and "–" on the third.

(v) He transformed $a+a-b-c$, $b-a+b-c$, and $c-a-b+c$ into $a+(a-b-c)$, $b-(a-b+c)$, and $c-(a+b-c)$ respectively. Then he tried another transformations like

$a+a-b-c=(a-c)+(a-b)$, $b-a+b-c=(b-c)-(a-b)$, $c-a-b+c=-(a-c)-(b-c)$, and said "How about this combination?" After that, he transformed the expression obtained at the stage (iv) into $\{(a-c)+(a-b)\}\log a+\{(b-c)-(a-b)\}\log b+\{-(a-c)-(b-c)\}\log c$, then into $(a-c)\{\log a-\log c\}+(b-c)\{\log b-\log c\}+(a-b)\{\log a-\log b\}$. He added "≥0" to this expression and said "so, this is positive." The solver closed this solving process by himself, which took 43 minutes.

2. 3 *The Outline of the T's Solving Process*

(i) After trying some numbers for *a*, *b*, *c* and checking whether the given inequality held, he listed up several ideas for a proof, and mentioned the idea to find C which satisfied A≥ C ≥B (here, A and B may refer to the left-side and right-side of the inequality respectively). Then he began to prove the two-letter case of the inequality. After he tried to apply the relation between arithmetic and geometric means and examined a graph of $y=a^x$, he began to consider the difference between the left-side and right-side in the two-letter case. He consequently proved it by transforming this difference into $a^{\frac{a+b}{2}} b^{\frac{a+b}{2}}\left(\left(\frac{a}{b}\right)^{\frac{a-b}{2}}-1\right)\geq 0$.

(ii) Returning to the three-letter case, he tried to apply the result of the two-letter case to a part of the given inequality, $b^b c^c$. Since a^a can be seen as $a^{\frac{a+b+c}{3}} a^{\frac{2a-b-c}{3}}$, he attempted to search for α satisfying $a^{\frac{2a-b-c}{3}} \geq (bc)^{\alpha}$ and $(bc)^{\alpha+\frac{b+c}{2}} \geq (bc)^{\frac{a+b+c}{3}}$ (Indeed, if such α were found, he could have showed $a^a b^b c^c \geq a^{\frac{a+b+c}{3}}(bc)^{\alpha}(bc)^{\frac{b+c}{2}} \geq (abc)^{\frac{a+b+c}{3}}$, which proved the given inequality). However, he gave up this search.

(iii) He tried to show $\frac{a^a b^b c^c}{(abc)^{\frac{a+b+c}{3}}} \geq 1$. This time, he searched for α which satisfied $a^{\frac{2a-b-c}{3}} \geq (bc)^{\alpha}$ and $b^{a+b-\frac{a+b+c}{3}} c^{a+c-\frac{a+b+c}{3}} \geq 1$. He derived the condition $a \geq \frac{b+a-2c}{3}$ from the latter requirement. Then he focused on one letter *b* to reduce the complexity, and searched for α which satisfied $a^{\frac{2a-b-c}{3}} c^{\frac{2c-a-b}{3}} \geq b^{\alpha}$ and $b^{a+b-\frac{a+b+c}{3}} \geq 1$.

(iv) He wrote new expressions saying "That may be what I wanted," and reached the following;

$$\left(\frac{a}{c}\right)^{\frac{a-c}{3}}\left[\frac{a^{\frac{a-b}{3}}}{c^{\frac{b-c}{3}}}\right] b^b \geq b^{\frac{a+b+c}{3}}.$$

He looked for α satisfying $\left(\frac{a}{c}\right)^{\frac{a-c}{3}}\left[\frac{a^{\frac{a-b}{3}}}{c^{\frac{b-c}{3}}}\right] \geq b^{\alpha}$ and $b^{\alpha} b^{b} \geq b^{\frac{a+b+c}{3}}$ (Finding such α is not necessary for the solution in fact). But he could not find such α, partly because of the mistakes in his calculation. The solving process, which took 101 minutes, was closed by the intervention of the researcher.

3. Impact of the Solutions of the Simpler Problems

3.1 *Impacts Observed in the S's Solution*

Here will be analyzed two stages, (iv) and (v), of the S's solution, which were directly related to his proof of the given inequality. Some activities in these stages are similar to or corresponding to the activities done before tackling the two-letter case. In spite of those similarities, however, there are differences between the activities before and after tackling the two-letter case. I would explore impact of the solution of the two-letter case on the solving process for the original problem, by considering such differences.

(1) Taking the logarithms of the both sides of the inequality had been done even before he tackled the two-letter case. But, in the stage (ii), he rejected the ideas of considering the difference between the cubes of the both sides and of considering $\frac{a^a b^b c^c}{(abc)^{\frac{a+b+c}{3}}}$, because these ideas were essentially the same as taking the logarithms. That is, he was not confident of the effectiveness of taking the logarithms. After he had proved the two letter-case, he immediately began to take the logarithms of the both sides of the three-letter case again and did not change this direction. This implies that the solution of the simpler problem had shown the validity of the idea of taking the logarithms of the both sides.

(2) The way of investigating the expressions such as $3a-(a+b+c)$ and $2a-(b+c)$ had changed after tackling the two-letter case. At the stage (i), his attention was paid only to whether each expression was positive or negative, and was not paid to relations among those expressions. He added the sign "O" to positive expressions and "×" to negative ones and said "This can't make it go well." He seemed to assume that $\log a$ was positive, and try to show that the difference between the left-side and right-side of the given inequality was positive based on the following facts; (a) both of $\log a$ and $2a-(b+c)$ were positive and their product was also positive; (b) similar facts worked for $\log b(2b-(a+c))$ and $\log c(2c-(a+b))$; (c) the difference between the left-side and right-side was expressed as the sum of these three terms.

At the stages (iv) and (v), he treated such expressions in the context of "exchange [the letters] and factorize it well, " and so investigated them relating them to each other. This attempt was clearly observed in his behavior that, after transforming some expressions into $a+(a-b-c)$, $b-(a-b+c)$, $c-(a+b-c)$ at (v), he pointed to three a's of $(a-b-c)$, $(a-b+c)$, and $(a+b-c)$ with his finger saying "a's are arranged well, but others are not." This idea occurred naturally, in the two-letter case, during transforming the expressions, because, in the two-letter case, $(a-b)$ and $(b-a)$

had occurred and it was easier to see their relation. So his attention to the relation among the expressions can be considered an impact of the solution of the simpler problem.

(3) The idea of gathering common factors, in relating to (2), was observed only after his solution of two-letter case. Before that, he tried to transform the difference between the left-side and right-side of the given inequality (after taking their logarithms) into a certain sum of positive terms. Emphasis was put on determining whether each appearing term was positive or negative. After the two-letter case, by contrary, he intended to factorize that difference, and emphasis was put on finding common factors in different terms. Although he investigated the same difference before and after the two-letter case, what he tried to find or construct in it had changed. His new intention can be seen an impact of the solution of the two-letter case.

(4) His attention to certain forms of expressions, e.g. $(a–b)$ and $(a–c)$, can be also considered an impact of the solution of the simpler problem. When he made a transformation like $a+a–b–c= (a–c)+(a–b)$, he said "So, I can use a very analogy with this." This transformation was done, however, without a sufficient prospect of a final solution, since he said "What can I get by approaching in such a way?" during this transformation. Only after he wrote $(a–c)\{\log a–\log c\}+ (b–c)\{\log b–\log c\}+$ as the transformation proceeded, he said "I've got it." This suggests that his previous utterance about utility of an analogy meant that he could then begin a transformation similar to the two-letter case. In other words, the transformation like $a+a–b–c =(a–c)+(a–b)$ was justified not because it could produce a proof of the given inequality, but because it could make it easier to relate the original and simpler problems and make it possible to proceed a transformation similar to the two-letter case. The solution of the simpler problem had presented a context where the factors that would play an essential role in the later activities could be supported when they occurred.

3.2 *Impacts Observed in the T's Solution*

Although the subject T did not reach a complete proof, he obtained the following expression during his solving process;

$$\left(\frac{a}{c}\right)^{\frac{a-c}{3}} \bullet \frac{a^{\frac{a-b}{3}}}{c^{\frac{b-c}{3}}} \bullet b^{b} \geq b^{\frac{a+b+c}{3}} \quad \cdots \ (*).$$

Dividing the both sides of this by $b^{\frac{a+b+c}{3}}$ and transforming the left-side can lead to a proof of the given inequality. Thus, this (*) means the considerable progress of T's solving process, and so the part of obtaining this would be analyzed here.

At the stage (iii), when T tried to find α such that $a^{\frac{2a-b-c}{3}} c^{\frac{2c-a-b}{3}} \geq b^a$ and $b^{a+b-\frac{a+b+c}{3}} \geq 1$, he said "Doing this breaks the attempt." But when he transformed the left-side of the former condition to $a^{\frac{(a-b)+(a-c)}{3}} c^{\frac{(c-a)+(c-b)}{3}}$, he said "No, it doesn't break." He wrote newly $\dfrac{a^{\frac{a-b+a-c}{3}} b^b}{c^{\frac{(a-c)+(a-c)}{3}}} \geq b^{\frac{a+b+c}{3}}$ saying "That may be what I wanted" and transformed the left-side of this expression into $\left(\dfrac{a}{c}\right)^{\frac{(a-c)+(a-c)}{3}} b^b$, which includes his mistakes. Correcting the mistakes in the exponents, he reached the expression (*). He said "Doing this breaks the attempt" in writing $a^{\frac{2a-b-c}{3}} c^{\frac{2c-a-b}{3}}$, but he said "It doesn't break" when he modified it into $a^{\frac{(a-b)+(a-c)}{3}} c^{\frac{(c-a)+(c-b)}{3}}$. This suggests that transforming the exponent $\dfrac{2a-b-c}{3}$ into $\dfrac{(a-b)+(a-c)}{3}$ was a clue to the expression (*). Since he immediately proceeded to $\left(\dfrac{a}{c}\right)^{\frac{(a-c)+(a-c)}{3}}$ saying "That may be what I wanted," he might say "It doesn't break" with such a transformation in his mind. Taking account of the fact that he had proved the two-letter case by making the form like $\left(\dfrac{a}{b}\right)^{\frac{a-b}{2}}$ and that that form of expressions had never appeared elsewhere, it can be said that the solution of the two-letter case might show the possibility and validity of the transformation into that form. Just before the end of the process, he pointed to $a^{\frac{a-b}{3}}$, $c^{\frac{b-c}{3}}$, $b^{\frac{b-c}{3}}$, $b^{\frac{a-b}{3}}$, which appeared as a result of a certain transformation, and said "They seem similar to..." This can also be considered to show his orientation to a similar transformation.

4. Importance of Exploration of the Original Problem Situation

As shown in the previous section, in the cases of the both subjects, the transformation of 2*a*–*b*–*c* into (*a*–*b*)+(*a*–*c*) and other similar ones were the important clues to the progresses of their solving processes. This appearing form (*a*–*b*), a difference of two letters, is certainly easy to be found in solving simpler problem. In fact, in the S's solving activities, this element of (*a*–*b*) was naturally generated by taking logarithms of the both sides of the two-letter case and ordering them with respect to log*a* and log*b*. In the T's solution, the terms $a^{\frac{a-b}{2}}$ and $b^{\frac{b-a}{2}}$ appeared through factorization of $a^a b^b - (ab)^{\frac{a+b}{2}}$ by the common factors $a^{\frac{a+b}{2}}$ and $b^{\frac{a+b}{2}}$, and they had the forms (*a*–*b*) and (*b*–*a*) in their exponents. The transformations could be furthered, in the processes of the both subjects, by interpreting this (*b*–*a*) as –(*a*–*b*). In this sense, the solution of the simper problem can be considered to have shown the validity of such forms of expressions.

Such forms as (a–b) can be appear in the three-letter case only when some expressions like $2a-b-c$ and $a+a-b-c$ are transformed appropriately. But the solution of the simpler case cannot give information about those appropriate transformations. Analyzing S's and T's processes with respect to this point, it can be noted that activities with such transformations had been done in other contexts.

The subject T subtracted $\frac{a+b+c}{3}$ from $\frac{b+c}{2}$ in order to check which was bigger, in the context of finding an appropriate α at the stage (ii), and tried to determine whether the numerator $b+c-2a$ of their difference was positive or negative. In doing that, he transformed it into $(b-a)+(c-a)$ and said that it was absolutely negative. The transformation necessary for the activities at the later stages did appear here. His utterance that $(b-a)+(c-a)$ was absolutely negative might be supported by the fact that $(b-a)<0$ and $(c-a)<0$, which implies his attention to these differences of the pairs of two letters. In the earlier part of (iii), he calculated the difference of the exponents $\frac{2a-b-c}{3} - \frac{a+b-2c}{3}$ to get $(a-b)+(c-b)$, and checked which of $(a-b)$ and $(b-c)$ was bigger. He had invented the transformation which would become necessary later, in his attempt to determine which exponent was bigger.

The subject S checked whether $3a-(a+b+c)$ or $2a-b-c$ was positive or negative at the stage (i), before tackling the simpler problem. At the stage (iv) (after solving the simpler problem), he searched for the common factors to factorize in the three-letter case and checked whether some expressions were positive or negative because, in the two-letter case, interpreting the negative term $(b-a)$ as $-(a-b)$ made a factorization possible;

> S (37:33): This is...at least...this [$a+a-b-c$] is positive, this [$b-a+b-c$] is undecided, this [$c-a-b+c$] is also undecided, ah, this [$c-a-b+c$] is negative...

It can be said that, in doing this, he made the differences of two letters and determined whether each expression was positive or negative based on positiveness or negativeness of those differences, just as T did. That is, S had paid attention to the form of differences of two letters in the context of checking whether some expressions were positive or negative.

Here, the expressions $2a-b-c$ and $a+a-b-c$ were generated through operations on the problem situation, i.e. the given inequality, and can be regarded as new elements of this situation. So, the fact that they could be transformed into $(a-b)+(a-c)$ etc. is new information about the problem situation. The above discussion in this section can be restated as follows; the information about the problem situation obtained by the activities which were not directly related to the final solution, played a critical role in applying the solution of the simpler problem to search for a solution of the original problem. This coincides with the discussions of some researchers (Terada, 1991; Tsukahara, 1991) that applying solutions of simpler problems requires understanding of original problems.

Indeed, in the S's solving process, the solution of the two-letter case provided him with the idea of factorization essential to the final solution. But the final solution of the original problem was not constructed by translating the solution of the two-letter case into the three-letter case. What he aimed at first according to the two-letter-case solution was the organization of the situation in the form of (expressions of used letters without log)×(expressions of used letters with log). This is reflected in that he made at stage (v) $a+(a-b-c)$, $b-(a-b+c)$, and $c-(a+b-c)$, all of which included similar forms like (a*b*c) (* is + or –). On the other hand, the organization in the final solution was the sum of the terms in the form of (difference between two letters)× (expressions of two letters with log). The latter form of organization was not found by examining various ways of organization referring to the two-letter-case solution (i.e. giving new senses to the two-letter-case solution), but by aiming at the former organization, investigating the relations among expressions without log, paying attention to differences between two letters like $(a-b)$, and transforming expressions based on those differences. In other words, it was found by his exploration of the problem situation aiming at the former organization. His report in the interview supports this;

> But during separating the letters, I noted there were two *a*'s, like *a* minus *a* minus *c*, so combine *a* and this, another one...Since there are two, so try to separate them, separate them further. I must have another $a-c$ elsewhere, so I've done that, then it worked well.

This utterance implies that the transformation into $(a-c)$ was continued based on a characteristic of the problem situation that two *a*'s existed in one term and on an attempt to treat them separately and combine them to other letters, rather than on an effort to make the form of $(a-c)$ or $(b-c)$ because the term $(a-b)$ became the common factor in the two-letter case. The final organization of the problem situation seems a result of such transformation. That is, the solution of the original problem was not attained by, in the original problem situation, searching elements which were needed to solve the simpler problem (see Polya, 1973, p. 111). During his activities with an attempt to make correspondence of the original problem with the simpler, he found new unexpected elements in the original situation, and importance of these elements was supported by the solution of the simpler problem. Organizing the original situation based on those elements, as a result, a structure of the situation different from the expected one occurred and it led to the solution of the original problem.

5. Utility of Simpler Problems and Giving Senses

To sum up the above discussion, according to the examples analyzed here, contributions of the solution of the simpler problem are suggesting some aspects to which solvers should pay attention in exploring the original problem situation and supporting importance of some elements in the situation (which may be obtained through activities not directly related to the final solution), rather than presenting the very procedure for the solution or the results available for it.

While the importance of selecting appropriate simpler problems has been emphasized in the previous research (e.g. Polya, 1973, pp. 52-53; Schoenfeld, 1985, pp. 84-96), little attention has been paid to how solvers use solutions of simpler problems to tackle the original problems. The above analysis shows, however, that it is not always simple to use those solutions to tackle the original problems. One of its reasons may be that it is a solvers' role to make sense of the solution of the simpler problem and decide how to apply that sense-making to the original problem situation. It is difficult to decide which may be better to make sense of the S's two-letter-case solution as (expression of the used letters without log)×(expression of the used letters with log) or as (the difference of the two letters)×(the difference of logs of the letters), referring only to the solution of the two-letter case. Like as utility of diagrams (Nunokawa, 1994a), senses given by the solvers are important factors in utility of simpler problems.

The above analysis also shows that failure of making-sense dose not necessarily mean the failure of using simpler problems. Even making-sense which was inappropriate to the final solution promoted exploring the problem situation and made it possible for the solvers to generate new information. If an appropriate making-sense cannot be determined uniquely, in using solutions of simpler problems, it seems important not only to translate procedures or results of simpler problems to original problems, but also to continue to explore the problem situation following information obtained by tentative senses of the simpler-case solutions.

6. Concluding Remarks

In this paper, the actual problem solving processes were analyzed and one aspect of the utility of simpler problems, that is, how solutions of the simpler problems can *indirectly* promote the solvers' exploration of the problem situation, was found. Taking account of this aspect, we can introduce "Using Simpler Problems" strategy, in the problem solving strategy instruction, in a little different way. The point emphasized in the introduction may be what kind of exploration can be continued according to solutions of simpler problems.

While using simpler problems can change solver's structures of a problem situation (Nunokawa, 1994b) in above-mentioned ways, making these simpler problems may be influenced by the solver's structure at that time. Interactions between used simpler problems and solver's structures are to be investigated in future research.

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