ISOMORPHISM BETWEEN JACOBI FORMS OF INDEX D_{2n+1} AND ELLIPTIC MODULAR FORMS OF LEVEL 2

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ABSTRACT. There are three aims in this paper: (i) We show an isomorphism between Jacobi forms of index D_{2n+1} (lattice index) and elliptic modular forms of level 2. (ii) We give an explicit formula of Fourier coefficients of Jacobi-Eisenstein series of index D_{2n+1}. (iii) We construct a holomorphic modular form of weight 3/2 of level 8 from the Zagier Eisenstein series $\mathscr{F}$ of weight 3/2 of level 4. Moreover, we show that the four functions E_2^*, η^3, θ^3 and $\mathscr{F}$ have essentially the same Hecke eigenvalue $1+p$ for any odd prime p, where E_2^* is the non-holomorphic Eisenstein series of weight 2, η is the Dedekind eta-function and θ is the usual theta function. This fact follows from a special case of the isomorphism of (i).

As an application, we give a formula for a sum of the numbers $r_3(n)$, where $r_3(n)$ is the number of representations of an integer $n \geq 0$ as a sum of 3 squares.

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1. INTRODUCTION

1.1. The isomorphism between Jacobi forms of *integer index* and a certain subspace of elliptic modular forms with level is given in [S-Z 88]. There are mainly three purposes in this present paper. The first purpose of this paper is to generalize this isomorphism from the case of Jacobi forms of integer index to the case of Jacobi forms of *index* D_r (r is odd). Here, D_r is one of the root lattices. More precisely, we will show that the

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space of Jacobi forms of weight $k+\frac{r+1}{2}$ of index D_r corresponds to certain subspace of elliptic modular forms of weight $2k$ of level 2 as Hecke modules (Theorems 1.1, 1.2 and 1.3). This isomorphism has been conjectured by A. Mocanu [Mo 19b].

The second purpose of this paper is to give an explicit formula of Fourier coefficients of Jacobi-Eisenstein series of index D_r (r is odd). This formula is written as linear combinations of Fourier coefficients of the Cohen Eisenstein series (Theorem 8.4).

The third purpose of this paper is to construct a holomorphic modular form of weight 3/2 of level 8 from the Zagier Eisenstein series $\mathscr{F}$ of weight 3/2 of level 4 (Theorem 1.5). Moreover, we will show that the four functions η^3, θ^3, $\mathscr{F}$ and E_2^* have essentially the same Hecke eigenvalue $1+p$ for any odd prime p, where η is the Dedekind eta-function, θ is the usual theta function and E_2^* is the non-holomorphic Eisenstein series of weight 2 (Theorems 1.4 and 1.5). As an application we give a formula between two arithmetic functions: the divisor sum and the number of representations of a given integer as a sum of 3 squares (Corollary 1.7).

We give also formulas of the map from Jacobi forms of index D_r (r is odd) to elliptic modular forms of level 2 (Theorems 1.8 and 1.9).

Let us explain more precisely these results.

1.2. **Isomorphisms.** We denote by J_{k,D_r} the space of Jacobi forms of weight k of index D_r (see §2.7 and §2.1 for the definition). Remark that Jacobi forms of lattice index are isomorphic to Jacobi forms of matrix index (see §2.1).

Let $M_k(N)$ (resp. $M_k^{new}(N)$) be the space of elliptic modular forms (resp. new forms) of weight $k\in\mathbb{N}$ with respect to $\Gamma_0(N)$. For $\epsilon\in\{+,-\}$ we put

$$M_k^{new,\epsilon}(N) := \{f\in M_k^{new}(N)\ :\ (f|_k\left(\begin{smallmatrix}0&-1\\N&0\end{smallmatrix}\right))(\tau) := f\left(\frac{-1}{N\tau}\right)(N^{1/2}\tau)^{-k} = \epsilon i^{-k} f(\tau)\}.$$

Remark that if $f\in M_k^{new,\epsilon}(N)$, then the completed L-function $L^*(f,s)$ of f satisfies $L^*(f,s)=\epsilon L^*(f,k-s)$ (see [S-Z 88, P. 136]).

It is shown in [S-Z 88, Main Thm.], that the isomorphism

$$J_{k+1,m}^{new}\cong M_{2k}^{new,-}(m)$$

holds for any natural numbers m and k. Here $J_{k+1,m}^{new}$ denotes the subspace of new forms of Jacobi forms of weight $k+1$ of index m.

In particular, the isomorphisms

$$J_{k+1,D_1}\ \cong\ J_{k+1,2}\ \cong\ M_{2k}^{new,-}(2)\oplus M_{2k}^{-}(1) \tag{1.1}$$

as Hecke modules holds, where $J_{k+1,2}$ denotes the space of Jacobi forms of weight $k+1$ of index 2.

A. Mocanu [Mo 19b] posed the following conjecture, which claims a generalization of (1.1). This conjecture is based on numerical examples of Euler factors of Jacobi forms in J_{k,D_r}.

Conjecture 1 ([Mo 19b, Conj. 3.30]). *For every $k \geq 2$, the following holds:*

$$\begin{aligned} J_{k+2,D_3} &\cong M_{2k}^{new,-}(2) \oplus M_{2k}^{+}(1), \\ J_{k+3,D_5} &\cong M_{2k}^{new,+}(2) \oplus M_{2k}^{-}(1), \\ J_{k+4,D_7} &\cong M_{2k}^{new,+}(2) \oplus M_{2k}^{+}(1) \end{aligned}$$

and these isomorphisms are Hecke equivalent.

It is known that if $r \equiv r' \bmod 8$ for odd natural numbers r and r', then the isomorphism $J_{k+\frac{r+1}{2},D_r} \cong J_{k+\frac{r'+1}{2},D_{r'}}$ as Hecke modules holds (see Corollary 2.8).

For $D \equiv 0, 1 \bmod 4$, we denote by $\left(\frac{D}{*}\right)$ the Kronecker symbol. For $D \in \{-4, 8, -8\}$, the Kronecker symbols satisfy

$$\left(\frac{-4}{r}\right) = (-1)^{\frac{r-1}{2}}, \ \left(\frac{8}{r}\right) = (-1)^{\frac{r^2-1}{8}}, \ \left(\frac{-8}{r}\right) = \left(\frac{-4}{r}\right)\left(\frac{8}{r}\right)$$

for odd integer r and $\left(\frac{-4}{r}\right) = \left(\frac{8}{r}\right) = \left(\frac{-8}{r}\right) = 0$ for even integer r. We define

$$\epsilon_2 := -\left(\frac{-8}{r}\right) \quad \text{and} \quad \epsilon_1 := -\left(\frac{-4}{r}\right).$$

The isomorphisms in Conjecture 1 are rewritten as

$$J_{k+\frac{r+1}{2},D_r} \cong M_{2k}^{new,\epsilon_2}(2) \oplus M_{2k}^{\epsilon_1}(1).$$

The main purpose of this paper is to show:

Theorem 1.1. *Conj. 1 is true.*

Theorem 1.1 will follow from Theorems 1.2 and 1.3.

To show Theorem 1.1, we first consider the composition of two maps:

$$M_{2k}(1) \to \left\{ \begin{array}{c} \text{Siegel modular forms of} \\ \text{weight } k + \frac{r+1}{2} \in 2\mathbb{N} \text{ of degree } r+1 \end{array} \right\} \to J_{k+\frac{r+1}{2},D_r},$$

where the first map is the Ikeda lift, while the second map gives a Fourier-Jacobi coefficient. We denote by "old forms in $J_{k+\frac{r+1}{2},D_r}$" the image of the above composition. We will show:

Theorem 1.2. *For $r = 1, 3, 5$ and 7, we have the decomposition*

$$J_{k+\frac{r+1}{2},D_r} = J_{k+\frac{r+1}{2},D_r}^{new} \oplus J_{k+\frac{r+1}{2},D_r}^{old}$$

such that the following isomorphisms as Hecke modules hold

$$\begin{aligned} J_{k+1,D_1}^{old} &\cong J_{k+3,D_5}^{old} \cong M_{2k}^{-}(1) = \begin{cases} \{0\}, & \text{if } k \text{ is even}, \\ M_{2k}(1), & \text{if } k \text{ is odd}, \end{cases} \\ J_{k+2,D_3}^{old} &\cong J_{k+4,D_7}^{old} \cong M_{2k}^{+}(1) = \begin{cases} M_{2k}(1), & \text{if } k \geq 2 \text{ is even}, \\ \{0\}, & \text{if } k \text{ is odd}. \end{cases} \end{aligned} \tag{1.2}$$

Here J^{new}_{k,D_r} denotes the orthogonal complement of J^{old}_{k,D_r} in J_{k,D_r} with respect to the Petersson scalar product of Jacobi forms (*see* [Mo 19a, P. 303], [Aj 15, P. 68] *or* [Mo 19b, P. 13] *for the definition of the Petersson scalar product*).

We will prove Theorem 1.2 in §3.

Remark that the dimension of space of Jacobi-Eisenstein series is less than or equal to 1 (see Lemma 4.1), and it is known that Jacobi-Eisenstein series is orthogonal to Jacobi cusp forms with respect to the Petersson scalar product (see [Mo 19a, Thm. 4.1]).

If the weight $k+\frac{r+1}{2}$ is odd, then the fact $J^{old}_{k+\frac{r+1}{2},D_r}=\{0\}$ follows from (1.2).

In the case $k=0$ in Theorem 1.2, it is known that $J_{2,D_3}=\{0\}$ and $J_{4,D_7}=\mathbb{C}E_{4,D_7}$ with a certain Jacobi form E_{4,D_7} (see [Mo 19b, Thm. 3.29] which is obtained by [B-S 23]). Such a Jacobi form E_{4,D_7} is also obtained as a Fourier-Jacobi coefficient of a Siegel theta series of weight 4 of degree 8.

We denote by $S_k(N)$ (resp. $S^{new}_k(k)$) the subspace of cusp forms in $M_k(N)$ (resp. $M^{new}_k(N)$). For $k\geq 2$, we have $M^{new}_{2k}(2)=S^{new}_{2k}(2)$ (see Lemma 4.3).

We denote by J^{cusp}_{k,D_r} the space of Jacobi cusp forms in J_{k,D_r} (see §2 for the definition). We put $J^{cusp,new}_{k,D_r}:=J^{cusp}_{k,D_r}\cap J^{new}_{k,D_r}$. We will show $J^{new}_{k+\frac{r+1}{2},D_r}\subset J^{cusp}_{k+\frac{r+1}{2},D_r}$ for $k\geq 2$ in Lemma 4.2. Thus $J^{cusp,new}_{k+\frac{r+1}{2},D_r}=J^{new}_{k+\frac{r+1}{2},D_r}$ for $k\geq 2$.

To show Theorem 1.1, we next consider the composition of two isomorphisms:

$$J^{cusp,new}_{k+\frac{r+1}{2},D_r}\cong\left\{\begin{array}{c}\text{certain modular forms}\\ \text{of weight } k+\frac{1}{2}\end{array}\right\}\cong S^{new,\epsilon_2}_{2k}(2). \tag{1.3}$$

The second isomorphism of this composition is an analogue of the Shimura correspondence which has been obtained in [U-Y 10] and in [Ya 14].

Let $S^{+}_{k+\frac{1}{2}}(8)$ be the Kohnen plus space of level 8 (see §2.4 for the definition). The space $S^{+}_{k+\frac{1}{2}}(8)$ has the decomposition: $S^{+}_{k+\frac{1}{2}}(8)=S^{+,-1}_{k+\frac{1}{2}}(8)\oplus S^{+,-5}_{k+\frac{1}{2}}(8)$ for odd k, and $S^{+}_{k+\frac{1}{2}}(8)=S^{+,-3}_{k+\frac{1}{2}}(8)\oplus S^{+,-7}_{k+\frac{1}{2}}(8)$ for even k (see §2.4, for the notation). Any form $g=\sum_n c_g(n)q^n$ in $S^{+,-r}_{k+\frac{1}{2}}(8)$ satisfies the condition:

$$c_g(n)=0 \text{ unless } n\equiv 0,4,-r \mod 8.$$

And $S^{+,-r}_{k+\frac{1}{2}}(8)$ is characterized by this condition. Although the space $S^{+,-r}_{k+\frac{1}{2}}(8)$ was not defined, but this space has been introduced in [U-Y 10, Prop. 4]. Let $S^{new,+,-r}_{k+\frac{1}{2}}(8)$ be the subspace of new forms in $S^{+,-r}_{k+\frac{1}{2}}(8)$ (see §6.3 for the definition of new forms in $S^{+}_{k+\frac{1}{2}}(8)$ which has been introduced in [U-Y 10]). The Shimura correspondence $S^{new,+,-r}_{k+\frac{1}{2}}(8)\cong S^{new,\epsilon_2}_{2k}(2)$ has been essentially shown in [U-Y 10, Thm. 1] (see Theorem 6.7).

On the other hand, the Shimura correspondence $\eta^{3r}M_{k+\frac{1-3r}{2}}(1)\cong S^{new,\epsilon_2}_{2k}(2)$ has been essentially shown in [Ya 14, Thm. 2] (see Theorem 5.3). Here η denotes the Dedekind eta-function $\eta(\tau):=q^{\frac{1}{24}}\prod_n(1-q^n)$, and where we put $q=e^{2\pi i\tau}$. Remark that we apply here the twisted Hecke operators on the space $\eta^{3r}M_{k+\frac{1-3r}{2}}(1)$ (see §2.6 for the definition of the twisted Hecke operators).

By combining two kinds of the composition (1.3), we show:

Theorem 1.3. *We have the following isomorphisms as Hecke modules. If $k \geq 2$ is even, then*

$$\begin{array}{ccccccccc} J^{cusp,new}_{k+1,D_1} & \cong & J^{cusp,new}_{k+2,D_3} & \cong & S^{new,+,-3}_{k+\frac{1}{2}}(8) & \cong & \eta^{21}M_{k-10}(1) & \cong & S^{new,-}_{2k}(2), \\ J^{cusp,new}_{k+3,D_5} & \cong & J^{cusp,new}_{k+4,D_7} & \cong & S^{new,+,-7}_{k+\frac{1}{2}}(8) & \cong & \eta^{9}M_{k-4}(1) & \cong & S^{new,+}_{2k}(2). \end{array}$$

If $k \geq 3$ is odd, then

$$\begin{array}{ccccccccc} J^{cusp,new}_{k+1,D_1} & \cong & J^{cusp,new}_{k+2,D_3} & \cong & S^{new,+,-1}_{k+\frac{1}{2}}(8) & \cong & \eta^{15}M_{k-7}(1) & \cong & S^{new,-}_{2k}(2), \\ J^{cusp,new}_{k+3,D_5} & \cong & J^{cusp,new}_{k+4,D_7} & \cong & S^{new,+,-5}_{k+\frac{1}{2}}(8) & \cong & \eta^{3}M_{k-1}(1) & \cong & S^{new,+}_{2k}(2). \end{array}$$

Theorem 1.3 will follow from Theorems 5.1 and 6.1.

1.3. **Fourier coefficients of Jacobi-Eisenstein series.** In Theorem 8.4 we give an explicit formula of Fourier coefficients of Jacobi-Eisenstein series $E_{k+\frac{r+1}{2},D_r}$ in $J_{k+\frac{r+1}{2},D_r}$.

Let $\mathscr{H}_k$ be the Cohen Eisenstein series of weight $k+\frac{1}{2}$ of level 4 (see §8). We construct a modular form $\mathscr{H}_k^*$ of weight $k+\frac{1}{2}$ of level 8 from $\mathscr{H}_k$. The form $\mathscr{H}_k^*$ is constructed by using $U_k(4)$-operator. The $U_k(4)$-operator has been introduced in [U-Y 10]. The Fourier coefficients of $\mathscr{H}_k^*$ are linear combinations of those of $\mathscr{H}_k$.

On the other hand, the Jacobi-Eisenstein series $E_{k+\frac{r+1}{2},D_r}$ corresponds to $\mathscr{H}_k^*$ (see Proposition 6.3). Thus the Fourier coefficients of $E_{k+\frac{r+1}{2},D_r}$ are expressed as linear combinations of those of $\mathscr{H}_k$.

1.4. **Modular forms of weight $3/2$ of level 8.** We will prove Conjecture 1 also for the case $k = 1$ (Theorem 1.4). In particular, we will show that the four functions η^3, θ^3, $\mathscr{F}$ and E_2^* have essentially the same Hecke eigenvalue $1+p$ for any odd prime p. Here we define $E_2^*(\tau) := 1 - 24\sum_{n\geq 1}\sigma(n)q^n - \frac{3}{\pi v}$, where $v = \mathrm{Im}(\tau)$, and where $\sigma(n)$ denotes the sum of divisors of n, and where $\theta(\tau) := \sum_{n\in\mathbb{Z}} q^{n^2}$, and where $\mathscr{F}$ denotes the Zagier Eisenstein series of weight 3/2 of level 4 (see §8 for the definition).

Let $M^{+,-r}_{k+\frac{1}{2}}(8)$ be a certain subspace of the Kohnen plus space $M^{+}_{k+\frac{1}{2}}(8)$ of weight $k+\frac{1}{2}$ of level 8 (see § 2.4 for the definition). We remark that $S^{+,-r}_{k+\frac{1}{2}}(8) = M^{+,-r}_{k+\frac{1}{2}}(8) \cap S^{+}_{k+\frac{1}{2}}(8)$. We show that a holomorphic modular form $E^{(8)}_{3/2} \in M^{+,-5}_{3/2}(8)$ of weight 3/2 of level 8 is obtained from θ^3 and also from $\mathscr{F}$ (see Theorem 1.5).

Theorem 1.4. *As an analogue of Theorem 1.3 for the case $k = 1$, we have*

$$\dim J_{2,D_1} = \dim J_{3,D_3} = \dim M^{+,-1}_{3/2}(8) = \dim \eta^{15}M_{-6}(1) = \dim M_2^-(2) = 0,$$

$$\dim J^{new}_{4,D_5} = \dim J^{cusp,new}_{5,D_7} = \dim M^{+,-5}_{3/2}(8) = \dim \eta^{3}M_{0}(1) = \dim M_2^+(2) = 1,$$

and we have the isomorphisms

$$\mathbb{C}E_{4,D_5} \cong \mathbb{C}\psi_{5,D_7} \cong \mathbb{C}E_{3/2}^{(8)} \cong \mathbb{C}\eta^3 \cong \mathbb{C}E_2^{(2)}$$

as Hecke modules. Here E_{4,D_5}*,* ψ_{5,D_7} *and* $E_{3/2}^{(8)}$ *are functions such that* $J_{4,D_5}^{new} = \mathbb{C}E_{4,D_5}$*,* $J_{5,D_7}^{cusp,new} = \mathbb{C}\psi_{5,D_7}$ *and* $M_{3/2}^{+,-5}(8) = \mathbb{C}E_{3/2}^{(8)}$*, and where* $M_2^+(2) = \mathbb{C}E_2^{(2)}$ *with* $E_2^{(2)}(\tau) := 2E_2^*(2\tau) - E_2^*(\tau)$*. Each of the three functions* E_{4,D_5}*,* $E_{3/2}^{(8)}$ *and* $E_2^{(2)}$ *is not Jacobi cusp form nor cusp form and* ψ_{5,D_7} *is a Jacobi cusp form. Remark that* $J_{4,D_5}^{new} \not\subset J_{4,D_5}^{cusp,new}$*.*

In particular, the five functions E_{4,D_5}*,* ψ_{5,D_7}*,* $E_{3/2}^{(8)}$*,* η^3 *and* $E_2^{(2)}$ *have the same Hecke eigenvalue* $1+p$ *for any odd prime* p*.*

We prove Theorem 1.4 in §7.2.

The Jacobi form $E_{4,D_5} \in J_{4,D_5}$ in Theorem 1.4 is determined up to a constant multiple. We quote E_{4,D_5} from [B-S 23] (See [Mo 19b, PP. 76–77]. See also Theorem 8.5 for an explicit formula of the Fourier coefficients).

The modular form $E_{3/2}^{(8)} \in M_{3/2}^{+,-5}(8)$ in Theorem 1.4 is determined up to a constant multiple. Since the constant term of $E_{3/2}^{(8)}$ is not 0, we can take $E_{3/2}^{(8)}$ as $E_{3/2}^{(8)}(\tau) = 1+O(q)$. In Theorem 1.5 we show that the modular form $E_{3/2}^{(8)}$ is obtained from θ^3 and also from the Zagier Eisenstein series $\mathscr{F}$ of weight 3/2. Due to Proposition 6.3, the Fourier coefficients of $E_{3/2}^{(8)}$ are essentially the same as those of E_{4,D_5}.

We now explain the Fourier coefficients of $E_{3/2}^{(8)}$. For $m \in \mathbb{N}$, we put

$$r_m(N) \;\; := \;\; \#\left\{(x_1,...,x_m) \in \mathbb{Z}^m \;:\; x_1^2 + \cdots + x_m^2 = N\right\}.$$

Remark that if $m \le 3$, then it holds $r_m(4N) = r_m(N)$ for any natural number N. Let $H(N)$ be the usual Hurwitz class number for discriminant $-N$ (see [Co 75, P. 273] for the definition. See also [Co 75, PP.284–285] for the table).

For any function f on $\mathfrak{H}$ such that $f(\tau+1) = f(\tau)$, we define the operator $U(4)$ by $f|U(4) := \frac{1}{4}\sum_{a \bmod 4} f\left(\frac{\tau+a}{4}\right)$. For any formal power series $f = \sum_{n\in\mathbb{Z}} a(n)q^n$, we define the operator $\wp_k$ by $f|\wp_k := \sum_{(-1)^k n \equiv 0,1 \bmod 4} a(n)q^n$. We put $U_k(4) := U(4)\wp_k$.

If $f(\tau) = \sum_{n=0}^{\infty} a(n)q^n$, then

$$(f|U_1(4))(\tau) \;\; = \;\; \sum_{\substack{n\ge 0 \\ n\equiv 0,3 \bmod 4}} a(4n)q^n.$$

Theorem 1.5. *The modular form* $E_{3/2}^{(8)} = 1 + O(q) \in M_{3/2}^{+,-5}(8)$ *has the expressions*

$$E_{3/2}^{(8)}(\tau) \;\; = \;\; \sum_{\substack{n\ge 0 \\ n\equiv 0,3 \bmod 4}} r_3(n)q^n \;\; = \;\; \sum_{\substack{n\ge 0 \\ n\equiv 0,3 \bmod 4}} r_3(4n)q^n \;\; = \;\; (\theta^3|U_1(4))(\tau).$$

Furthermore, the modular form $E^{(8)}_{3/2}$ has also the expressions

$$E^{(8)}_{3/2}(\tau) \;=\; 12 \sum_{\substack{n\geq 0\\ n\equiv 0,3 \bmod 4}} (H(4n)-2H(n))q^n \;=\; 12(\mathscr{F}|U_1(4)-2\mathscr{F})(\tau),$$

where $\mathscr{F}$ is the non-holomorphic modular form of weight $3/2$ introduced in [Za 75].

The modular form $E^{(8)}_{3/2}$ has the property $E^{(8)}_{3/2}|U_1(4) = E^{(8)}_{3/2}$. In particular, we have

$$r_3(N) \;=\; 12(H(4N)-2H(N))$$

for any natural number N. Here we define $H(N):=0$ for $-N\equiv 2,3 \bmod 4$.

We prove Theorem 1.5 in §8.3.

The following formula has been obtained in [Co 75, P. 274].

Corollary 1.6. *For any $N\in\mathbb{N}$, we have*

$$r_3(N) \;=\; 12H(Df_1)\left(1-\left(\frac{8}{D}\right)\right), \tag{1.4}$$

where $-N = Df^2$ with a fundamental discriminant $D<0$ and $f\in\frac{1}{2}\mathbb{N}$, and where $f=2^e f_1$ with an odd integer f_1 and an integer $e\geq -1$.

1.5. **Application to arithmetic functions.** Since $\theta E^{(8)}_{3/2}\in M_2(8)$ and since $M_2(8) = \mathbb{C}E^{(2)}_2(\tau)\oplus\mathbb{C}E^{(2)}_2(2\tau)\oplus\mathbb{C}E^{(2)}_2(4\tau)$, by comparing the Fourier coefficients of $\theta E^{(8)}_{3/2}$ and those of the basis of $M_2(8)$, we have

$$\theta(\tau)E^{(8)}_{3/2}(\tau) \;=\; \frac{1}{12}E^{(2)}_2(\tau)-\frac{1}{12}E^{(2)}_2(2\tau)+E^{(2)}_2(4\tau).$$

Corollary 1.7. *For any $N\in\mathbb{N}$, we have*

$$\sum_{\substack{0\leq s\leq\sqrt{N}\\ N-s^2\equiv 0,3 \bmod 4}} \tilde{\delta}_s r_3(N-s^2) \;=\; \sigma(N)-3\,\sigma\left(\frac{N}{2}\right)+14\,\sigma\left(\frac{N}{4}\right)-24\,\sigma\left(\frac{N}{8}\right), \tag{1.5}$$

where we define $\tilde{\delta}_s := \frac{1}{2}$ or 1 according as $s=0$ or $\neq 0$. Here we regard $\sigma(M)$ as 0 if $M\notin\mathbb{N}$. In particular, if N is an odd integer, then

$$\sigma(N) \;=\; \sum_{\substack{0\leq s\leq\sqrt{N}\\ N-s^2\equiv 0,3 \bmod 4}} \tilde{\delta}_s r_3(N-s^2). \tag{1.6}$$

For example, $\sigma(1) = r_3(0)$, $\sigma(3) = \frac{1}{2}r_3(3)$, $\sigma(5) = r_3(4)$, $\sigma(7) = r_3(3)$, $\sigma(9) = r_3(8)+r_3(0)$, $\sigma(11) = \frac{1}{2}r_3(11)$, $\sigma(13) = r_3(12)+r_3(4)$, $\sigma(15) = r_3(11)$, and so on.

We remark that a similar formula of (1.5) follows from the facts $\theta^4\in M_2(4)$ and $M_2(4) = \mathbb{C}E^{(2)}_2(\tau)\oplus\mathbb{C}E^{(2)}_2(2\tau)$. Namely, $\theta(\tau)^4 = \frac{1}{3}E^{(2)}_2(\tau)+\frac{3}{2}E^{(2)}_2(2\tau)$. Thus we have the well-known formula $r_4(n) = 8\sigma(n) - 32\sigma(n/4)$. Since $r_4(n) = r_3(n) + 2\sum_{s>0} r_3(n-s^2)$, we have

$$(1.7)\qquad \sum_{0\le s\le\sqrt{N}} \tilde{\delta}_s r_3(N-s^2) \;=\; \frac{1}{2} r_4(N) \;=\; 4\sigma(N) - 16\,\sigma\left(\frac{N}{4}\right) \quad \text{for any } N\in\mathbb{N}.$$

In particular, we have

$$(1.8)\qquad \sigma(N) \;=\; \frac{1}{4}\sum_{0\le s\le\sqrt{N}} \tilde{\delta}_s r_3(N-s^2) \quad \text{for } N\not\equiv 0 \mod 4.$$

Note that the number of terms of s in (1.5) (resp. (1.6)) is less than that in (1.7) (resp. (1.8)).

1.6. **Maps from Jacobi forms to elliptic modular forms.** We set $M_2^{new,\epsilon_2}(2) := M_2^{\epsilon_2}(2)$. In §9 we will prove the formula of maps from $J_{k+\frac{r+1}{2},D_r}$ to $M_{2k}^{new,\epsilon_2}(2) \oplus M_{2k}^{\epsilon_1}(1)$ which are compatible with the action of Hecke operators. These maps are given as follows.

Theorem 1.8. *Assume that $k+\frac{r+1}{2}$ is an even integer. For $\phi \in J_{k+\frac{r+1}{2},D_r}$ we set*

$$A(N) \;:=\; \begin{cases} c(n',r'), & \text{if } N = 8(n'-\beta(r')) \text{ and } N\equiv 0,4 \mod 8,\\ 2c(n',r'), & \text{if } N = 8(n'-\beta(r')) \text{ and } N\equiv -r \mod 8,\end{cases}$$

where $c(n',r')$ denotes the (n',r')-th Fourier coefficient of ϕ for $(n',r') \in \mathbb{Z}\times D_r^\sharp$ (see §2.7 for the notations of β and $D_r^\sharp$).

Remark that $\sum_{N\equiv 0,4,-r \bmod 8} A(N)q^N \in M^{+,-r}_{k+\frac{1}{2}}(8)$.

Then, for any fundamental discriminant $(-1)^k d_0$ such that $d_0>0$ and $d_0\equiv 0 \mod 4$, the map S_{d_0} defined by

$$S_{d_0}(\phi) \;:=\; \frac{A(0)L(1-k,\left(\frac{(-1)^k d_0}{*}\right))}{2(1+\left(\frac{8}{r}\right)2^k)} + \sum_{n=1}^{\infty}\left(\sum_{d|n}\left(\frac{(-1)^k d_0}{d}\right) d^{k-1} A\!\left(\frac{n^2}{d^2}d_0\right)\right) q^n$$

is a linear map

$$S_{d_0} \;:\; J_{k+\frac{r+1}{2},D_r} \to M_{2k}^{new,\epsilon_2}(2)\oplus M_{2k}^{\epsilon_1}(1).$$

The map S_{d_0} maps $J^{cusp}_{k+\frac{r+1}{2},D_r}$ to $S_{2k}^{new,\epsilon_2}(2)\oplus S_{2k}^{\epsilon_1}(1)$, $J^{new}_{k+\frac{r+1}{2},D_r}$ to $M_{2k}^{new,\epsilon_2}(2)$ and $J^{old}_{k+\frac{r+1}{2},D_r}$ to $M_{2k}^{\epsilon_1}(1)$ and commutes with the action of Hecke operators.

We remark that if $k+\frac{r+1}{2}$ is odd, then $J_{k+\frac{r+1}{2},D_r} = J^{cusp,new}_{k+\frac{r+1}{2},D_r}$ (See Theorem 1.2 and Lemma 4.1).

Theorem 1.9. *Assume that $k+\frac{r+1}{2}$ is an odd integer. For $\phi \in J^{cusp,new}_{k+\frac{r+1}{2},D_r}$ and for $N = 8n'-r$, we set $A(N) := c(n',v_1)$ (see § 4 for the notation v_1) and where $c(n',r')$ denotes the (n',r')-th Fourier coefficient of ϕ for $(n',r') \in \mathbb{Z}\times D_r^\sharp$.*

We remark that $\sum_{n'\in\mathbb{Z}} c(n',v_1)q^{(8n'-r)/8} \in \eta^{24-3r}M_{k-12+\frac{3r+1}{2}}(1)$.

For any fundamental discriminant $(-1)^{k-1}d_0$ such that $d_0>0$ and $d_0 \equiv -r \mod 8$, we define

$$S_{d_0}(\phi) \;:=\; \sum_{n=1}^{\infty}\left(\left(\left(\frac{8}{r}\right)2^{k-1}\right)^{e_2}\left(\frac{-4}{n_1}\right)\sum_{d|n_1}\left(\frac{(-1)^k d_0}{d}\right)d^{k-1}A\left(\frac{n_1^2}{d^2}d_0\right)\right)q^n,$$

where n_1 is an odd integer and e_2 is a natural number determined by $n=2^{e_2}n_1$. Then S_{d_0} is a linear map

$$S_{d_0} \;:\; J^{cusp,new}_{k+\frac{r+1}{2},D_r} \to S^{new,\epsilon_2}_{2k}(2).$$

Moreover, the map S_{d_0} commutes with the action of Hecke operators.

We prove Theorems 1.8 and 1.9 in §9.

We remark that similar maps to those in Theorems 1.8 and 1.9 for the case D_1 have been obtained in the context of Jacobi forms of matrix index in [Br 06].

The present paper is organized as follows; in Section 2, definitions of Jacobi forms, modular forms of half-integral weight and Hecke operators are explained. In Section 3, through Ikeda lifting and Fourier-Jacobi expansion, we prove Theorem 1.2. In Section 4, some basic properties of Jacobi forms of index D_r are explained. In Section 5, we show an isomorphism between elliptic modular forms and Jacobi forms of index D_r of odd weight, while we show this isomorphism for Jacobi forms of even weight in Section 6. In Section 7, we prove Theorems 1.1 and 1.4. In Section 8, an explicit formula of Fourier coefficient of Jacobi-Eisenstein series in J_{k,D_r} is given and a proof of Theorem 1.5 is given. In Section 9, Theorems 1.8 and 1.9 are proved.

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2. Jacobi forms of lattice index

2.1. **Basic properties.** We denote by $\mathfrak{H}$ the Poincare upper half space. We put $e(x) := e^{2\pi i x}$. We write $q := e(\tau)$ for $\tau \in \mathfrak{H}$. For each $z \in \mathbb{C}$, we take the argument $-\pi < \arg(z) \le \pi$ and set $z^{m/2} := (z^{1/2})^m$ for $m \in \mathbb{Z}$.

Let L be a free $\mathbb{Z}$-module of finite rank n equipped with a $\mathbb{Z}$-valued symmetric binary form β. In this paper we assume that β is positive-definite. Such a pair $\underline{L} = (L, \beta)$ is called an *integral lattice* of rank n. By abuse of language, we define $\beta(x) := \frac{1}{2}\beta(x,x)$. Sometimes we use $x \mapsto \beta(x)$, instead of $(x,y) \mapsto \beta(x,y)$. If $\beta(x) \in \mathbb{Z}$ for any $x \in L$, then $\underline{L}$ is called an *even* integral lattice.

For example, if M is a positive-definite half-integral symmetric matrix of size n, then

$$(\mathbb{Z}^n, (x,y) \mapsto x(2M)^t y)$$

is an even integral lattice.

We put

$$L^\sharp \;:=\; \{x \in L \otimes_{\mathbb{Z}} \mathbb{Q} \;:\; \beta(x,y) \in \mathbb{Z} \text{ for any } y \in L\}.$$

We fix a $\mathbb{Z}$-basis $\{e_i\}_i$ of L, and we identify $\mathbb{C}^n$ with $L \otimes_{\mathbb{Z}} \mathbb{C}$ by $(x_1, ..., x_n) \mapsto \sum_i x_i e_i$.

Let k be an integer

Definition 2.1. *A holomorphic function $\phi(\tau, z)$ on $\mathfrak{H} \times \mathbb{C}^n$ is called a Jacobi form of weight k and index $\underline{L}$, if ϕ satisfies the following three conditions:*

(i) *For all $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$, the function ϕ satisfies*

$$(\phi|_{k,\underline{L}}A)(\tau,z) \;:=\; \phi\left(\frac{a\tau+b}{c\tau+d}, \frac{1}{c\tau+d}z\right) e\left(\frac{-c\beta(z)}{c\tau+d}\right)(c\tau+d)^{-k} \;=\; \phi(\tau,z).$$

(ii) *For all λ, $\mu \in L$, the function ϕ satisfies*

$$(\phi|_{k,\underline{L}}(\lambda,\mu))(\tau,z) \;:=\; \phi(\tau, z+\lambda\tau+\mu)e(\tau\beta(\lambda)+\beta(\lambda,z)) \;=\; \phi(\tau,z).$$

(iii) *The function $\phi(\tau, z)$ has a Fourier expansion of the form*

$$\phi(\tau,z) \;=\; \sum_{n' \in \mathbb{Z}} \sum_{r' \in L^\sharp} c(n',r') e(n'\tau + \beta(r',z)),$$

where $c(n',r') = 0$ unless $n' - \beta(r') \geq 0$.

The complex numbers $c(n',r')$ are called the (n',r')-th Fourier coefficients of ϕ.

If $c(n',r') = 0$ unless $n' - \beta(r') > 0$ in (iii)*, then ϕ is called a Jacobi cusp form.*

We denote by $J_{k,\underline{L}}$ (resp. $J_{k,\underline{L}}^{cusp}$) the space of all Jacobi forms (resp. Jacobi cusp forms) of weight k of index $\underline{L}$.

In Definition 2.1, if the lattice $\underline{L} = (L, \beta)$ coincides with $(\mathbb{Z}^n, x \mapsto \beta(x) = xM^t x)$ with a positive-definite half-integral symmetric matrix M of size n, then ϕ is called a Jacobi form of weight k of index M (see [Zi 89]). We denote by $J_{k,M}$ the space of all Jacobi forms of weight k of index M.

Let $\underline{L} = (L, \beta)$ be an integral lattice with a $\mathbb{Z}$-basis $\{e_1, ..., e_n\}$ of L. The matrix $2M = (\beta(e_i, e_j))_{i,j}$ is called the *Gram matrix* of β with respect to $\{e_1, ..., e_n\}$. Then $L \cong \mathbb{Z}^n$, $L^\sharp \cong (2M)^{-1}\mathbb{Z}^n$ and $L^\sharp/L \cong \mathbb{Z}^n/(2M\mathbb{Z}^n)$. Moreover, if $\underline{L}$ is an even integral lattice, then we have $J_{k,\underline{L}} \cong J_{k,M}$ (see [B-S 23, §1]).

2.2. **Discriminant modules.** For an even integral lattice $\underline{L} = (L, \beta)$, *the discriminant module* $D_{\underline{L}}$ of $\underline{L}$ is a pair defined by

$$D_{\underline{L}} \quad := \quad \left(L^\sharp/L, x + L \mapsto \beta(x) + \mathbb{Z}\right)$$

(see [B-S 23, §2]). We remark that $L^\sharp/L$ is a finite abelian group and $x+L \mapsto \beta(x)+\mathbb{Z}$ is the associated quadratic form.

Two discriminant modules $D_{\underline{L_i}} = (L_i^\sharp/L_i, x + L_i \to \beta_i(x) + \mathbb{Z})$ $(i = 1, 2)$ are called isomorphic, if there exists a homomorphism $j : L_1^\sharp/L_1 \to L_2^\sharp/L_2$ which satisfies $\beta_2(j(x))+\mathbb{Z} = \beta_1(x) + \mathbb{Z}$.

2.3. **Theta functions.** For $x \in L^\sharp$, we put

$$\theta_{\underline{L},x}(\tau, z) \quad := \quad \sum_{y \in x+L} e(\tau\beta(y) + \beta(y, z)) \quad ((\tau, z) \in \mathfrak{H} \times \mathbb{C}^n).$$

The following formulas are well known.

Lemma 2.2. *For $r_i \in L^\sharp$, we have*

$$\theta_{\underline{L},r_i}(\tau + 1, z) \quad = \quad e(\beta(r_i))\theta_{\underline{L},r_i}(\tau, z)$$

and

$$\begin{aligned} &\theta_{\underline{L},r_i}(-\tau^{-1}, \tau^{-1}z) \\ &\quad = \left(\frac{\tau}{i}\right)^{n/2} e(\tau^{-1}\beta(z))\frac{1}{\sqrt{|L^\sharp/L|}} \sum_{r_j \in L^\sharp/L} e(-\beta(r_i, r_j))\theta_{\underline{L},r_j}(\tau, z). \end{aligned} \tag{2.1}$$

(*See* [Ar 98, page 100], [Aj 15, Theorem 2.3.3], [Boy 11, Cor. 3.34], [B-S 23, page 16], *etc.*).

Lemma 2.3. *Let $\phi \in J_{k,\underline{L}}$ be a Jacobi form. Then, ϕ has the decomposition:*

$$\phi(\tau, z) \quad = \quad \sum_{x \in L^\sharp/L} h_x(\tau)\theta_{\underline{L},x}(\tau, z) \ \textit{with}\ h_x(\tau) \quad = \quad \sum_{n' \in \mathbb{Z}} c(n', x)e^{2\pi i(n'-\beta(x))\tau}.$$

Since $c(n', x) = c(n' + \beta(x, y) + \beta(y), x + y)$ for any $y \in L$, the function h_x is determined by x modulo L and by $n' - \beta(x)$. Thus the above decomposition of ϕ is well-defined.

Theorem 2.4 ([B-S 23, Thm. 2.5])**.** *Let $\underline{L_1}$ and $\underline{L_2}$ be two positive-definite even lattices. Assume that $j : D_{\underline{L_1}} \xrightarrow{\cong} D_{\underline{L_2}}$ is an isomorphism of the discriminant modules. Then the map*

$$I_j \quad : \quad J_{k+\lfloor \frac{n_1}{2} \rfloor, \underline{L_1}} \to J_{k+\lfloor \frac{n_2}{2} \rfloor, \underline{L_2}}$$

given by

$$\phi(\tau, z) = \sum_{x \in L_1^\sharp/L_1} h_x(\tau)\theta_{\underline{L_1},x}(\tau, z) \mapsto I_j(\phi)(\tau, z') = \sum_{x \in L_1^\sharp/L_1} h_x(\tau)\theta_{\underline{L_2},j(x)}(\tau, z')$$

is an isomorphism as $\mathbb{C}$-vector spaces, where n_i $(i = 1, 2)$ is the rank of L_i.

It is shown in [Aj 15, Thm. 4.2.4] that the isomorphism in the above theorem is also compatible with the action of Hecke operators (see Theorem 2.7 below).

2.4. **Modular forms of half-integral weight.** We denote by $\mathfrak{G}$ the set of all pairs $(A, \omega(\tau))$, where $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is an element of the connected component $\mathrm{GL}^+(2,\mathbb{R})$ of $\mathrm{GL}(2,\mathbb{R})$ and ω is a holomorphic function on the upper-half plane $\mathfrak{H}$ which satisfies

$$|\omega(\tau)|^2 \;=\; (\det A)^{-\frac{1}{2}}|c\tau + d|.$$

The group operation of $\mathfrak{G}$ is given by $(A_1, \omega_1(\tau))) \cdot (A_2, \omega_2(\tau)) := (A_1A_2, \omega_1(A_2\tau)\omega_2(\tau))$.

We put

$$\theta(\tau) \;:=\; 1 + \sum_{m\in\mathbb{Z}} e(m^2\tau).$$

There exists an injective homomorphism $\Gamma_0(4) \to \mathfrak{G}$ given by

$$A \mapsto (A, j(A,\tau))$$

with

$$j(A,\tau) \;:=\; \frac{\theta(A\tau)}{\theta(\tau)}, \quad \left(A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)\right).$$

It is known

$$j(A,\tau) \;=\; \left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{-1/2}(c\tau+d)^{1/2}, \quad \left(A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)\right),$$

where $\left(\frac{c}{d}\right)$ is the Kronecker symbol (cf. [Mi 89, P. 194]).

Let N be a natural number such that $4|N$. For $k \in \mathbb{Z}$, we denote by $M_{k+\frac{1}{2}}(N)$ the vector space of holomorphic functions f, which satisfy the conditions:

$$(f|_{k+\frac{1}{2}}A)(\tau) \;:=\; j(A,\tau)^{-2k-1}f(A\tau) \;=\; f(\tau)$$

for any $A \in \Gamma_0(N)$ and f^2 is an elliptic modular form of weight $2k+1$ of level N with the character $\left(\frac{-4}{d}\right)$. In this paper we call $f \in M_{k+\frac{1}{2}}(N)$ a modular form of weight $k+\frac{1}{2}$ with level N. A function $f \in M_{k+\frac{1}{2}}(N)$ is a cusp form, if f^2 is a cusp form. We denote by $S_{k+\frac{1}{2}}(N)$ the space of all cusp forms in $M_{k+\frac{1}{2}}(N)$.

We put the Kohnen plus-space of level 8:

$$M^+_{k+\frac{1}{2}}(8) \;:=\; \left\{ g = \sum_n c_g(n)q^n \in M_{k+\frac{1}{2}}(8) \;:\; c_g(n) = 0 \text{ unless } n \equiv 0, (-1)^k \mod 4 \right\}.$$

For $k \in \mathbb{Z}$ and for $r \in \{1,3,5,7\}$, such that $(-1)^k \equiv -r \mod 4$, we define

$$M^{+,-r}_{k+\frac{1}{2}}(8) \;:=\; \left\{ g = \sum_n c_g(n)q^n \in M_{k+\frac{1}{2}}(8) \;:\; c_g(n) = 0 \text{ unless } n \equiv 0, 4, -r \mod 8 \right\}.$$

For example, we have $\theta \in M^{+,-7}_{1/2}(8)$.

Remark $M^{+,-r}_{k+\frac{1}{2}}(8) \subset M^{+}_{k+\frac{1}{2}}(8)$. It follows from [U-Y 10, Prop. 4] that

$$M^{+}_{k+\frac{1}{2}}(8) = \begin{cases} M^{+,-1}_{k+\frac{1}{2}}(8) \oplus M^{+,-5}_{k+\frac{1}{2}}(8), & \text{if } k \text{ is odd,} \\ M^{+,-3}_{k+\frac{1}{2}}(8) \oplus M^{+,-7}_{k+\frac{1}{2}}(8), & \text{if } k \text{ is even.} \end{cases}$$

We put $S^{+}_{k+\frac{1}{2}}(8) := M^{+}_{k+\frac{1}{2}}(8) \cap S_{k+\frac{1}{2}}(8)$ and $S^{+,-r}_{k+\frac{1}{2}}(8) := M^{+,-r}_{k+\frac{1}{2}}(8) \cap S_{k+\frac{1}{2}}(8)$. Then, we have

$$S^{+}_{k+\frac{1}{2}}(8) = \begin{cases} S^{+,-1}_{k+\frac{1}{2}}(8) \oplus S^{+,-5}_{k+\frac{1}{2}}(8), & \text{if } k \text{ is odd,} \\ S^{+,-3}_{k+\frac{1}{2}}(8) \oplus S^{+,-7}_{k+\frac{1}{2}}(8), & \text{if } k \text{ is even.} \end{cases} \tag{2.2}$$

2.5. **Modular forms of half-integral weight of η-type.** We set

$$\mathrm{Mp}(2,\mathbb{Z}) := \left\{ (A, \omega(\tau)) \; : \; A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}), \omega(\tau)^2 = c\tau + d \right\}.$$

We regard $\mathrm{Mp}(2,\mathbb{Z})$ as a subgroup of $\mathfrak{G}$. We quote some facts about $\mathrm{Mp}(2,\mathbb{Z})$ from [B-S 23]. The group $\mathrm{Mp}(2,\mathbb{Z})$ is a two-fold central extension of $\mathrm{SL}(2,\mathbb{Z})$:

$$1 \to \{\pm 1\} \to \mathrm{Mp}(2,\mathbb{Z}) \to \mathrm{SL}(2,\mathbb{Z}) \to 1.$$

Two elements $\tilde{T} := \left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), 1\right)$ and $\tilde{S} := \left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \sqrt{\tau}\right)$ generates $\mathrm{Mp}(2,\mathbb{Z})$. We have $\tilde{S}^2 = (\tilde{S}\tilde{T})^3 = \left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \sqrt{-1}\right)$. And $\tilde{S}$ has the order 8 in $\mathrm{Mp}(2,\mathbb{Z})$.

It is known that the group of linear characters of $\mathrm{Mp}(2,\mathbb{Z})$ is the cyclic group of order 24 generated by ε, where

$$\varepsilon((A,\omega)) := \frac{\eta(A\tau)}{\eta(\tau)\omega(\tau)} \quad ((A,\omega) \in \mathrm{Mp}(2,\mathbb{Z})).$$

It is known that the definition of ε does not depend on the choice of $\tau \in \mathfrak{H}$. We have $\varepsilon(\tilde{T}) = e(1/24)$ and $\varepsilon(\tilde{S}) = \varepsilon(\tilde{T})^{-3} = e(-1/8)$. For the detail of the group of linear characters of $\mathrm{Mp}(2,\mathbb{Z})$, see [B-S 23, Prop. 1.1].

For $k \in \frac{1}{2}\mathbb{Z}$ and for $s \in \mathbb{Z}$ $(0 \le s \le 23)$, we denote by $M_{k+\frac{s}{2}}(1, \varepsilon^s)$ the vector space of holomorphic functions f, which satisfy the conditions:

$$f(A\tau) = \varepsilon((A,\omega))^s \omega(\tau)^{2k+s} f(\tau)$$

for any $(A,\omega) \in \mathrm{Mp}(2,\mathbb{Z})$ and f^{24} is an elliptic modular form of weight $24k + 12s$ of level 1. Since $\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), -1\right) \in \mathrm{Mp}(2,\mathbb{Z})$ and since $\varepsilon\left(\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), -1\right)\right) = -1$, if such non-zero function f exists, then $2k + 2s \equiv 0 \mod 2$. Hence, $\dim M_{k+\frac{s}{2}}(1, \varepsilon^s) = 0$ unless $k \in \mathbb{Z}$.

Lemma 2.5. *For $0 \le s \le 23$ and for $k \in \mathbb{Z}$, we have*

$$M_{k+\frac{s}{2}}(1, \varepsilon^s) = \eta^s M_k(1).$$

Proof. Let $\Delta := \eta^{24}$ be the Ramanujan Δ-function. Then, by transformation formula of $f \in M_{k+s/2}(1,\varepsilon^s)$ and by q-expansion of f, we have $\eta^{24-s}M_{k+s/2}(1,\varepsilon^s) \subset \Delta M_k(1)$. Thus $M_{k+s/2}(1,\varepsilon^s) \subset \eta^s M_k(1)$. The opposite inclusion $\eta^s M_k(1) \subset M_{k+s/2}(1,\varepsilon^s)$ is obvious. □

Following [Ya 14, P. 6], in this paper we call $f \in M_{k+\frac{s}{2}}(1,\varepsilon^s)$ a modular form of weight $k+\frac{s}{2}$ of η-type. In this paper we treat only the cases $s = 3, 9, 15$ and 21.

2.6. **Hecke operators.** Assume $k \in \mathbb{Z}$. Let $f \in M_{k+\frac{1}{2}}(4)$. We take the Fourier expansion $f(\tau) = \sum_{n=0}^{\infty} c(n)q^n$. For any odd prime p, the Hecke operator $T(p^2)$ is defined by

$$(f|_{k+\frac{1}{2}}T(p^2))(\tau) \ := \ \sum_{n=1}^{\infty}\left\{c(p^2n) + \left(\frac{(-1)^k n}{p}\right)p^{k-1}c(n) + p^{2k-1}c(n/p^2)\right\}q^n.$$

Here, we regard $c(n/p^2)$ as 0, if n/p^2 is not an integer.

Let $f \in \eta^r M_{k+\frac{1}{2}-\frac{r}{2}}(1)$ ($r = 3, 9, 15, 21$). The form f is a modular form of weight $k+1/2$ of η-type. The form f has the Fourier expansion $f(\tau) = \sum_{n=1}^{\infty} c(n)q^{n/8}$. Remark that $c(n) = 0$ unless $n \equiv r/3 \mod 8$. For any odd prime p, we introduce the *twisted* Hecke operator $\tilde{T}(p^2)$ which is defined by

$$(f|_{k+\frac{1}{2}}\tilde{T}(p^2))(\tau) \ := \ \left(\frac{-4}{p}\right)\sum_{n=1}^{\infty}\left\{c(p^2n) + \left(\frac{(-1)^k n}{p}\right)p^{k-1}c(n) + p^{2k-1}c(n/p^2)\right\}q^{n/8}.$$

Here, we regard $c(n/p^2)$ as 0, if n/p^2 is not an integer. Remark that the definition of $\tilde{T}(p^2)$ is different from the definition of T_{p^2} in [Ya 14, Prop. 11]. To show the isomorphism between $\eta^r M_{k+\frac{1-r}{2}}(1)$ and a certain space of Jacobi forms of odd weight as Hecke module (see Prop. 5.2), in this paper we apply $\tilde{T}(p^2)$ for $\eta^r M_{k+\frac{1-r}{2}}(1)$.

We introduce Hecke operators $T^J(p^2)$ acting on the space of Jacobi forms of index $\underline{L}$. The following lemma follows from [Mu 89, §4].

Lemma 2.6. *Let $\phi \in J_{k,\underline{L}}$. We take the Fourier expansion of ϕ:*

$$\phi(\tau,z) \ = \ \sum_{n'\in\mathbb{Z}}\sum_{r'\in L^\sharp} c(n',r')e(n'\tau + \beta(r',z)).$$

Let m be the rank of L. Assume $p \nmid (2[L^\sharp/L])$, and we put

$$(\phi|_{k,\underline{L}}T^J(p^2))(\tau,z) \ := \ \sum_{n'\in\mathbb{Z}}\sum_{r'\in L^\sharp} c^*(n',r')e(n'\tau + \beta(r',z)),$$

where

$$\begin{aligned} c^*(n',r') &:= c(p^2n',pr') \\ &+p^{k-m-2}c(n',r')\begin{cases}\left(\left(\frac{-1}{p}\right)p\right)^{\frac{m}{2}}\left(\frac{[L^\sharp/L]}{p}\right)\delta(n',r'), & \text{if } m\equiv 0 \mod 2,\\ \left(\left(\frac{-1}{p}\right)p\right)^{\frac{m+1}{2}}\left(\frac{2[L^\sharp/L]}{p}\right)\left(\frac{n'-\beta(r')}{p}\right), & \text{if } m\equiv 1 \mod 2,\end{cases} \\ &+p^{2k-m-2}\sum_{\lambda\in L/pL} c\left(\frac{1}{p^2}(n'-\beta(r',\lambda)+\beta(\lambda)),\frac{1}{p}(r'-\lambda)\right).\end{aligned}$$

Here, in the summation $\sum_{\lambda\in L/pL}$, *λ runs through a set of representatives of L/pL and there is only one $\lambda\in L/pL$ such that $\frac{1}{p}(r'-\lambda)\in L^\sharp$. We regard $c(n',r')=0$ if $(n',r')\notin\mathbb{Z}\times L^\sharp$. It means that the above summation* $\sum_{\lambda\in L/pL}$ *consists of at most one term. And here we denote*

$$\delta(n',r') := \begin{cases} p-1, & \text{if } n'-\beta(r')\equiv 0 \mod p,\\ -1, & \text{otherwise.}\end{cases}$$

Then, under the above definition, we have $\phi|T^J(p^2)\in J_{k,\underline{L}}$. Furthermore, if ϕ is a Jacobi cusp form, then $\phi|T^J(p^2)$ is also a Jacobi cusp form.

In this paper we call ϕ a Hecke eigenform, if ϕ is an eigenfunction with respect to $T^J(p)$ for any prime p such that $p \not| (2[L^\sharp/L])$.

We remark that if m is odd, then the definition of $T^J(p^2)$ coincides with $T(l)$ $(l=p)$ in [Aj 15, Thm. 2.6.1].

Theorem 2.7 ([Aj 15, Thm. 4.2.4]). *We assume $n_1\equiv n_2\equiv 1 \mod 2$. Then, the isomorphism*

$$I_j \;:\; J_{k+\lfloor\frac{n_1}{2}\rfloor,\underline{L_1}} \xrightarrow{\cong} J_{k+\lfloor\frac{n_2}{2}\rfloor,\underline{L_2}}$$

in Theorem 2.4 is compatible with the action of Hecke operators $T^J(p)$:

$$I_j(\phi|_{k+\lfloor\frac{n_1}{2}\rfloor}T^J(p)) \;=\; (I_j(\phi))|_{k+\lfloor\frac{n_2}{2}\rfloor}T^J(p)$$

for any $\phi\in J_{k+\lfloor\frac{n_1}{2}\rfloor,\underline{L_1}}$.

2.7. **Jacobi forms of index D_r.** We quote the definition of lattice $D_r := \underline{D_r}$ mainly from [Mo 19b, §3.3]. The lattice D_r is defined by

$$D_r \;:=\; \{(x_1,...,x_r)\in\mathbb{Z}^r \;:\; x_1+\cdots+x_r\in 2\mathbb{Z}\}$$

which is equipped with the Euclidian bilinear form:

$$\beta((x_1,...,x_r),(y_1,...,y_r)) \;=\; x_1y_1+\cdots+x_ry_r.$$

We have

$$D_r^\sharp \quad = \quad \left\{ (x_1, ..., x_r) \in \mathbb{Z}^r \cup \left(\frac{1}{2} + \mathbb{Z}\right)^r \right\}.$$

We can take a set of representatives of $D_r^\sharp / D_r$ as

$$D_r^\sharp / D_r \quad = \quad \begin{cases} \left\{0, e_r, \frac{e_1+\cdots+e_r}{2}, \frac{e_1+\cdots+e_{r-1}-e_r}{2}\right\}, & \text{if } r > 1, \\ \left\{0, e_1, \frac{1}{2}e_1, -\frac{1}{2}e_1\right\}, & \text{if } r = 1, \end{cases}$$

where $\{e_i\}_i$ denotes the standard basis of $\mathbb{Z}^r$.

We assume that r is an odd integer. Then $D_r^\sharp / D_r \cong \mathbb{Z}/4\mathbb{Z}$ as $\mathbb{Z}$-module. The discriminant module D_{D_r} of D_r satisfies

$$D_{D_r} \quad \cong \quad \left(\mathbb{Z}/4\mathbb{Z}, x + 4\mathbb{Z} \mapsto \frac{rx^2}{8} + \mathbb{Z} \right).$$

Therefore the discriminant module D_{D_r} is determined by r mod 8. By virtue of Theorems 2.4 and 2.7, we have the following corollary.

Corollary 2.8 ([B-S 23, Thm. 2.5], [Aj 15, Thm. 4.2.4])**.** *If* $r_1 \equiv r_2 \mod 8$ *and if* r_1 *is odd, then*

$$J_{k+\frac{r_1+1}{2}, D_{r_1}} \cong J_{k+\frac{r_2+1}{2}, D_{r_2}}$$

as Hecke modules.

Therefore it is sufficient to consider the cases $r = 1, 3, 5$ and 7.

We remark that the space of Jacobi forms of lattice D_r index is isomorphic to the space of the Jacobi forms of matrix M_r index:

$$J_{k,D_r} \quad \cong \quad J_{k,M_r}, \tag{2.3}$$

where $M_1 = 2$, $M_3 = \begin{pmatrix} 1 & u & u \\ u & 1 & u \\ u & u & 1 \end{pmatrix}$, $M_5 = \begin{pmatrix} 1 & u & 0 & 0 & u \\ u & 1 & u & 0 & 0 \\ 0 & u & 1 & u & 0 \\ 0 & 0 & u & 1 & u \\ u & 0 & 0 & u & 1 \end{pmatrix}$, and $M_7 = \begin{pmatrix} 1 & u & 0 & 0 & 0 & 0 & u \\ u & 1 & u & 0 & 0 & 0 & 0 \\ 0 & u & 1 & u & 0 & 0 & 0 \\ 0 & 0 & u & 1 & u & 0 & 0 \\ 0 & 0 & 0 & u & 1 & u & 0 \\ 0 & 0 & 0 & 0 & u & 1 & u \\ u & 0 & 0 & 0 & 0 & u & 1 \end{pmatrix}$, and where $u = 1/2$. Here $2M_r$ is the Gram matrix of D_r with respect to the basis $\{2e_1\}$ for $r = 1$ and $\{e_1 + e_2, e_2 + e_3, ..., e_{r-1} + e_r, e_r + e_1\}$ for $r = 3, 5$ and 7. We can check that $\det(2M_r) = 4$ for $r = 1, 3, 5$ and 7.

3. Ikeda lifting

We denote by $M_k(\mathrm{Sp}_n(\mathbb{Z}))$ the space of Siegel modular forms of weight k with respect to $\mathrm{Sp}_n(\mathbb{Z})$. Here $\mathrm{Sp}_n(\mathbb{Z})$ is the symplectic group of size $2n$. We denote by $S_k(\mathrm{Sp}_n(\mathbb{Z}))$ the subspace of all Siegel cusp forms in $M_k(\mathrm{Sp}_n(\mathbb{Z}))$.

Theorem 3.1 (Duke-Imamoḡlu [B-K 00], Ikeda [Ik 01])**.** *Let k and n be positive integers, such that $k+n$ is even. Then, there exists an injective linear map*

$$I_n \ : \ S_{2k}(1) \ \rightarrow \ S_{k+n}(\mathrm{Sp}_{2n}(\mathbb{Z})),$$

in which I_n maps Hecke eigenforms to Hecke eigenforms. In the case $n=1$, I_1 coincides with the Saito-Kurokawa lifting. For the details, the reader refer to [Ik 01].

We denote by L_n^* the set of all half-integral symmetric matrices of size n. We denote by $\mathfrak{H}_n$ the Siegel upper half space of size n. We call a matrix $M \in L_n^*$ *maximal*, if M satisfies the condition: if $A^{-1}M\,{}^tA^{-1} \in L_n^*$ with $A \in \mathrm{GL}_n(\mathbb{R}) \cap M_n(\mathbb{Z})$, then $A \in \mathrm{GL}_n(\mathbb{Z})$. Here $M_n(\mathbb{Z})$ denotes the set of all $n \times n$ matrices with entries in $\mathbb{Z}$.

Lemma 3.2. *Let $I_n(f) \in S_{k+n}(\mathrm{Sp}_{2n}(\mathbb{Z}))$ be the image of $f \in S_{2k}(1)$ by I_n. We take the Fourier-Jacobi expansion of $I_n(f)$:*

$$I_n(f)\left(\begin{pmatrix} \tau & z \\ {}^tz & \omega \end{pmatrix}\right) \ = \sum_{M \in L_{2n-1}^*} \phi_M(\tau, z) e(M\omega), \quad (\tau \in \mathfrak{H}, z \in \mathbb{C}^{2n-1}, \omega \in \mathfrak{H}_{2n-1}).$$

Then, ϕ_M is a Jacobi cusp form of weight $k+n$ of matrix M index. Moreover, if f is a Hecke eigenform and if M is maximal, then ϕ_M is a Hecke eigenform.

Proof. It is well known that ϕ_M is a Jacobi cusp form of weight k of index M for any $M \in L_{2n-1}^*$ (see [Zi 89, Introduction]). The fact that ϕ_M is a Hecke eigenform follows from the facts:

(i) Jacobi-Eisenstein series is a Hecke eigenform (see [Aj 15, Thm 3.3.18]),

(ii) M-th Fourier-Jacobi coefficient of Siegel-Eisenstein series is a Jacobi-Eisenstein series, if M is maximal (see [Boe 83, Satz 7]), and

(iii) Ikeda lift $I_n(f)$ inherits some relations among Fourier coefficients of Siegel-Eisenstein series, and since M-th Fourier-Jacobi coefficients of Siegel-Eisenstein series is a Hecke eigenform, we can conclude that ϕ_M is also a Hecke eigenform.

We omit the details (see [Ha 11], for example). □

Lemma 3.3. *Assume $2n = r+1$. If $2M_r$ is the Gram matrix of D_r $(r = 1,3,5,7)$, and f is a Hecke eigenform, then ϕ_{M_r} in Lemma 3.2 is not identically zero. Therefore ϕ_{M_r} is a non-zero Hecke eigenform in $J_{k+\frac{r+1}{2},M_r}^{cusp}$.*

Proof. Since there exists $u_1, u_2 \in D_r^\#$ such that $\beta(u_1) = 0$ and $\beta(u_2) = \frac{1}{2}$, the set $\{8(n' - \beta(u)) \ : \ n' \in \mathbb{Z}, u \in D_r^\#\}$ contains all positive integers which are divisible by 4. It is shown in [Ko 80, Page 260] that there exists a fundamental discriminant $D \equiv 0 \mod 4$ such that $c_h(|D|) \neq 0$, where $h \in S_{k+n+1/2}^+(4)$ corresponds to f by the Shimura correspondence and where $c_h(|D|)$ is the $|D|$-th Fourier coefficient of h. Then, by the construction of Fourier coefficient of the Ikeda lift (cf. [Ik 01, P. 642]), ϕ_{M_r} is not identically zero. Thus this lemma follows from Lemma 3.2. □

Therefore, by virtue of the isomorphism (2.3), we obtain the following proposition.

Proposition 3.4. *If* $k \equiv \frac{r+1}{2} \mod 2$ *and if* $k \geq 2$*, then there exists an injective linear map* $\mathcal{I}$ *which is obtained by the composition of the Ikeda lift and the Fourier-Jacobi expansion:*

$$\mathcal{I} \ : \ M_{2k}^{\epsilon_1}(1) \to J_{k+\frac{r+1}{2},D_r}.$$

The map $\mathcal{I}$ *maps Eisenstein series to Jacobi Eisenstein series,* $S_{2k}^{\epsilon_1}(1)$ *to* $J^{cusp}_{k+\frac{r+1}{2},D_r}$ *and commutes with the action of Hecke operators.*

Proof. Recall $\epsilon_1 = -\left(\frac{-4}{r}\right)$. The condition $k \equiv \frac{r+1}{2} \mod 2$ is equivalent to $(-1)^k = (-1)^{\frac{r+1}{2}} = \epsilon_1$. Thus we have $M_{2k}(1) = M_{2k}^{\epsilon_1}(1)$ under the assumption in this proposition.

An Eisenstein series in $M_{2k}^{\epsilon_1}(1)$ corresponds to a Siegel-Eisenstein series by the Ikeda lift. And M-th Fourier Jacobi coefficient of a Siegel-Eisenstein series is a Jacobi-Eisenstein series, if M is maximal. Thus the map $\mathcal{I}$ maps Eisenstein series in $M_{2k}^{\epsilon_1}(1)$ to Jacobi-Eisenstein series in $J_{k+\frac{r+1}{2},D_r}$.

The rest of the proof of this proposition follows from Lemma 3.3. □

We now prove Theorem 1.2.

Proof of Theorem 1.2. The vector space $J^{old}_{k+\frac{r+1}{2},D_r}$ denotes the image of the map $\mathcal{I}$ in Proposition 3.4 and $J^{new}_{k+\frac{r+1}{2},D_r}$ is the orthogonal complement of $J^{old}_{k+\frac{r+1}{2},D_r}$ in $J_{k+\frac{r+1}{2},D_r}$ with respect to the Petersson scalar product. Thus $J_{k+\frac{r+1}{2},D_r} = J^{new}_{k+\frac{r+1}{2},D_r} \oplus J^{old}_{k+\frac{r+1}{2},D_r}$ and the fact $J^{old}_{k+\frac{r+1}{2},D_r} \cong M_{2k}^{\epsilon_1}(1)$ as Hecke modules follows from Proposition 3.4. □

4. Theta decomposition of Jacobi forms of index D_r

The lattice $\underline{D_r} = (D_r, \beta)$ for $r = 1, 3, 5$ and 7 has been explained in §2.7. We put

$$v_0 = 0,\ v_2 = e_r,\ v_1 = \frac{e_1 + \cdots + e_r}{2},\ v_3 = \frac{e_1 + \cdots + e_{r-1} - e_r}{2}$$

for $r = 3, 5, 7$ and put $v_j = \frac{j}{2} e_1$ $(j = 0, 1, 2, 3)$ for $r = 1$. Then, $\{v_0, v_2, v_1, v_3\}$ is a set of representatives of $D_r^\sharp/D_r$. We have

$$\beta(v_0) = 0,\ \beta(v_2) = \frac{1}{2},\ \beta(v_1) = \beta(v_3) = \frac{r}{8}.$$

Remark that $v_j + D_r \mapsto j + 4\mathbb{Z}$ gives the isomorphism $D_r^\sharp/D_r \cong \mathbb{Z}/(4\mathbb{Z})$ as $\mathbb{Z}$-modules. Furthermore, the associated quadratic form of $D_r^\sharp/D_r$ is given by

$$v_j + D_r \ \ \mapsto \ \ \beta(v_j) + \mathbb{Z} = \frac{j^2 r}{8} + \mathbb{Z}.$$

We remark $\beta(v_i, v_j) + \mathbb{Z} = \frac{ijr}{4} + \mathbb{Z}$.

For simplicity, we put

$$\theta_{r,j} \ \ := \ \ \theta_{D_r,v_j}.$$

and put $J_{k,D_r} := J_{k,\underline{D_r}}$. By virtue of Lemma 2.2, we have

$$\theta_{r,j}(\tau+1,z) \;=\; e(j^2r/8)\theta_{r,j}(\tau,z) \tag{4.1}$$

and

$$\theta_{r,j}(-\tau^{-1},\tau^{-1}z) \;=\; \frac{1}{2}\tau^{r/2}e(-r/8)e(\tau^{-1}\beta(z))\sum_{i=0}^{3} e(-ijr/4)\theta_{r,i}(\tau,z). \tag{4.2}$$

Let $\phi \in J_{k,D_r}$. We have the decomposition

$$\phi(\tau,z) \;=\; \sum_{j=0}^{3} h_j(\tau)\theta_{r,j}(\tau,z).$$

The functions h_j are uniquely determined by ϕ (see Lemma 2.3).

Since $2v_0, 2v_2, v_1+v_3 \in D_r$, and due to the definition of $\theta_{r,j}$, we have $\theta_{r,j}(\tau,-z) = \theta_{r,j}(\tau,z)$ for $j=0,2$ and $\theta_{r,1}(\tau,-z) = \theta_{r,3}(\tau,z)$. Since $\phi(\tau,-z) = (-1)^k\phi(\tau,z)$, we have

$$\begin{cases} h_1(\tau) = h_3(\tau), & \text{if } k \text{ is even,} \\ h_0(\tau) = h_2(\tau) = 0 \text{ and } h_1(\tau) = -h_3(\tau), & \text{if } k \text{ is odd.} \end{cases}$$

Thus

$$\begin{aligned} &\phi(\tau,z) \\ &\quad= \begin{cases} h_0(\tau)\theta_{r,0}(\tau,z) + h_2(\tau)\theta_{r,2}(\tau,z) + h_1(\tau)(\theta_{r,1}(\tau,z)+\theta_{r,1}(\tau,-z)), & \text{if } k \text{ is even,} \\ h_1(\tau)(\theta_{r,1}(\tau,z) - \theta_{r,1}(\tau,-z)), & \text{if } k \text{ is odd.} \end{cases} \end{aligned}$$

Remark that the constant term of ϕ is the constant term of h_0.

Lemma 4.1. *Let the notation be as above. A Jacobi form $\phi \in J_{k,D_r}$ is a Jacobi cusp form, if and only if the constant term of $h_0(\tau)$ is 0.*

We have $\dim J_{k,D_r} - \dim J_{k,D_r}^{cusp} \le 1$. In particular, if k is odd, then $J_{k,D_r} = J_{k,D_r}^{cusp}$.

Proof. Let $(n',r') \in \mathbb{Z} \times D_r^\sharp$. If $n' - \beta(r') = 0$, then $\beta(r') \in \mathbb{Z}$ and $r' \in D_r$.

Let $c(n',r')$ denotes the (n',r')-th Fourier coefficient of ϕ. For $(n_i',r_i') \in \mathbb{Z}\times D_r^\sharp$ $(i=1,2)$, if $n_1' - \beta(r_1') = n_2' - \beta(r_2')$ and if $r_1' \equiv r_2' \mod D_r$, then $c(n_1',r_1') = c(n_2',r_2')$.

Thus, if $c(0,v_0) = 0$, then ϕ is a Jacobi cusp form. On the other hand, $c(0,v_0)$ is the constant term of $h_0(\tau)$. Therefore ϕ is a Jacobi cusp form, if and only if the constant term of h_0 is 0.

Thus, for any integer k, the inequality $\dim J_{k,D_r} - \dim J_{k,D_r}^{cusp} \le 1$ follows.

If k is odd, then $h_0(\tau) = 0$ for any $\phi \in J_{k,D_r}$. Hence we have $J_{k,D_r} = J_{k,D_r}^{cusp}$. □

It is known that if $k + \frac{r+1}{2} \equiv 0 \mod 2$ with $k \ge 2$, then there exists a Jacobi-Eisenstein series $E_{k+\frac{r+1}{2},D_r} \in J_{k+\frac{r+1}{2},D_r}$ (see [Mo 19a, Thm. 4.1]). In this case we have $\dim J_{k+\frac{r+1}{2},D_r} - \dim J_{k+\frac{r+1}{2},D_r}^{cusp} = 1$.

Lemma 4.2. *For $k \geq 2$, we have $J^{new}_{k+\frac{r+1}{2},D_r} \subset J^{cusp}_{k+\frac{r+1}{2},D_r}$. In particular, we have $J^{cusp,new}_{k+\frac{r+1}{2},D_r} = J^{new}_{k+\frac{r+1}{2},D_r}$ for $k \geq 2$.*

Proof. If $k + \frac{r+1}{2} \equiv 1 \bmod 2$, then $J_{k+\frac{r+1}{2},D_r} = J^{cusp}_{k+\frac{r+1}{2},D_r}$. Thus $J^{new}_{k+\frac{r+1}{2},D_r} \subset J^{cusp}_{k+\frac{r+1}{2},D_r}$ follows.

We assume $k + \frac{r+1}{2} \equiv 0 \bmod 2$. Due to Proposition 3.4, we have a $E_{k+\frac{r+1}{2},D_r} \in J^{old}_{k+\frac{r+1}{2},D_r}$. Let $\phi \in J^{new}_{k+\frac{r+1}{2},D_r}$. If the constant term of ϕ is not 0, then ϕ can not be orthogonal to $E_{k+\frac{r+1}{2},D_r}$ with respect to the Petersson scalar product. It is contradiction to the fact $\phi \in J^{new}_{k+\frac{r+1}{2},D_r}$. Hence, the constant term of ϕ is 0 and $\phi \in J^{cusp}_{k+\frac{r+1}{2},D_r}$. ☐

Lemma 4.3. *For $k \geq 2$, we have $M^{new}_{2k}(2) \subset S_{2k}(2)$. In particular, we have $S^{new}_{2k}(2) = M^{new}_{2k}(2)$ for $k \geq 2$.*

Proof. It is known that $\dim M_{2k}(2) - \dim S_{2k}(2) = 2$ for $k \geq 2$. Let $E^{(1)}_{2k}$ be an Eisenstein series of weight $2k$ with respect to $\mathrm{SL}(2,\mathbb{Z})$. Functions $E^{(1)}_{2k}(\tau)$ and $E^{(1)}_{2k}(2\tau)$ are linearly independent. For $u, v \in \mathbb{C}$, if $f(\tau) := uE^{(1)}_{2k}(\tau) + vE^{(1)}_{2k}(2\tau)$ belongs to $S_{2k}(2)$, then the constant term of f is 0. Thus $v = -u$. For odd prime p, the p-th Fourier coefficient of f is $(1+p^{2k-1})u$ up to a constant multiple. Since f is a cusp form, f satisfies the Hecke bound $(1+p^{2k-1})u = O(p^k)$. Then, $u = v = 0$. We have $M_{2k}(2) = \mathbb{C}E^{(1)}_{2k}(\tau) \oplus \mathbb{C}E^{(1)}_{2k}(2\tau) \oplus S_{2k}(2)$. Since $\mathbb{C}E^{(1)}_{2k}(\tau) \oplus \mathbb{C}E^{(1)}_{2k}(2\tau)$ is a subspace of old forms, new forms are orthogonal to $\mathbb{C}E^{(1)}_{2k}(\tau) \oplus \mathbb{C}E^{(1)}_{2k}(2\tau)$. We conclude $M^{new}_{2k}(2) \subset S_{2k}(2)$. ☐

5. Jacobi forms of odd weight

In §5 and §6, we will consider the cases that weight $k + \frac{r+1}{2}$ of Jacobi forms is even and odd separately.

In this section we assume that $k + \frac{r+1}{2}$ is an odd integer. The purpose of this section is to show:

Theorem 5.1. *For $r \in \{1, 3, 5, 7\}$ and for odd integer $k + \frac{r+1}{2}$ with $k \geq 2$, we have*

$$J^{cusp}_{k+\frac{r+1}{2},D_r} \quad \cong \quad \eta^{24-3r} M_{k-12+\frac{3r+1}{2}}(1) \quad \cong \quad S^{new,\epsilon_2}_{2k}(2)$$

as Hecke modules, where we recall $\epsilon_2 = -\left(\frac{-8}{r}\right) = -1$ or $= 1$ according as $r = 1$, 3 or $= 5$, 7. Here we remark that we apply the twisted Hecke operators $\tilde{T}(p^2)$ on $\eta^{24-3r} M_{k-12+\frac{3r+1}{2}}(1)$ (see §2.6).

Theorem 5.1 follows from Proposition 5.2 and Corollary 5.4 (Theorem 5.3) below.

For $\phi \in J^{cusp}_{k+\frac{r+1}{2},D_r}$, we write

$$\phi(\tau, z) \quad = \quad h_1(\tau)(\theta_{r,1}(\tau, z) - \theta_{r,1}(\tau, -z)) \tag{5.1}$$

(see §4 for the notation).

Proposition 5.2. *For $r \in \{1,3,5,7\}$, if $k+\frac{r+1}{2}$ is an odd integer, then the map which is defined by*

$$\mathcal{J}_{r,k}^{odd} \ : \ \phi(\tau,z) \mapsto h_1(\tau)$$

gives an isomorphism

$$\mathcal{J}_{r,k}^{odd} \ : \ J_{k+\frac{r+1}{2},D_r}^{cusp} \cong \eta^{3(8-r)} M_{k-12+\frac{3r+1}{2}}(1)$$

as Hecke modules. It is

$$\mathcal{J}_{r,k}^{odd}(\phi|T^J(p)) \ = \ \mathcal{J}_{r,k}^{odd}(\phi)|\tilde{T}(p^2) \tag{5.2}$$

for any odd prime p.

Proof. It follows from Lemma 2.5 that $\eta^{3(8-r)} M_{k-12+\frac{3r+1}{2}}(1) = M_{k+\frac{1}{2}}(1,\varepsilon^{24-3r})$.

First we shall show $\mathcal{J}_{r,k}^{odd}(\phi)(=h_1) \in M_{k+\frac{1}{2}}(1,\varepsilon^{24-3r})$. By using identities (5.1), (4.1), (4.2) and transformation formulas of ϕ, we have

$$h_1(\tau+1) \ = \ e(-r/8)h_1(\tau)$$

and

$$h_1(-\tau^{-1}) \ = \ \tau^{k+1/2}e(3r/8)h_1(\tau).$$

Since $\varepsilon(\tilde{T}) = e(1/24)$ and $\varepsilon(\tilde{S}) = e(-1/8)$ (see §2.5 for the notation), we obtain identities

$$h_1(\tau+1) \ = \ \varepsilon(\tilde{T})^{24-3r}h_1(\tau), \tag{5.3}$$

$$h_1(-\tau^{-1}) \ = \ \varepsilon(\tilde{S})^{24-3r}\sqrt{\tau}^{2k+1}h_1(\tau). \tag{5.4}$$

Since $\tilde{T}$ and $\tilde{S}$ generates $\mathrm{Mp}(2,\mathbb{Z})$, the fact $h_1 \in M_{k+\frac{1}{2}}(1,\varepsilon^{24-3r})$ follows from the above identities (5.3) and (5.4). It is not difficult to see that the map $\mathcal{J}_{r,k}^{odd}$ is injective.

Next we shall show that $\mathcal{J}_{r,k}^{odd}$ is surjective. Let $h_1 \in M_{k+\frac{1}{2}}(1,\varepsilon^{24-3r})$. We put $\phi(\tau,z) = h_1(\tau)(\theta_{r,1}(\tau,z) - \theta_{r,1}(\tau,-z))$. Then, due to the transformation formula of h_1 and that of $\theta_{r,j}$ $(j=0,1,2,3)$ (see (5.3), (5.4), (4.1) and (4.2)), the form ϕ satisfies the transformation formula (i) and (ii) in Definition 2.1 as weight $k+\frac{r+1}{2}$ of index D_r. Since $\theta_{r,1}$ satisfies the condition (iii) in Definition 2.1, also ϕ satisfies the condition. Therefore $\phi \in J_{k+\frac{r+1}{2},D_r}$. It follows form Lemma 4.1 that $J_{k+\frac{r+1}{2},D_r} = J_{k+\frac{r+1}{2},D_r}^{cusp}$. Thus $\mathcal{J}_{r,k}^{odd}$ is surjective.

Finally, we shall show the identity (5.2). We take the Fourier expansion of $\phi \in J_{k+\frac{r+1}{2},D_r}^{cusp}$:

$$\phi(\tau,z) \ = \ \sum_{n'\in\mathbb{Z}} \sum_{r'\in D_r^\sharp} C(n',r')e(n'\tau + \beta(r',z)).$$

Then,

$$h_1(\tau) \ = \ (\mathcal{J}_{r,k}^{odd}(\phi))(\tau) \ = \ \sum_{n'\in\mathbb{Z}} C(n',v_1)e\left(\left(n'-\frac{r}{8}\right)\tau\right),$$

where $v_1 = \frac{e_1+\cdots+e_r}{2}$ has been defined in §4. Remark $\beta(v_1) = r/8$.

For $n' \in \mathbb{Z}$, we write $A(8n'-r) := C(n', v_1)$. Then, $h_1(\tau) = \sum_{m\in\mathbb{Z}} A(m) q^{m/8}$.

Let p be an odd prime. By definition of $\tilde{T}(p^2)$ (see §2.6), we have

$$(h_1|_{k+\frac{1}{2}}\tilde{T}(p^2))(\tau) \quad = \quad \sum_{m\in\mathbb{Z}} A^*(m) q^{m/8},$$

where

$$A^*(m) \quad = \quad \left(\frac{-4}{p}\right)\left\{A(p^2m) + \left(\frac{(-1)^k m}{p}\right)p^{k-1}A(m) + p^{2k-1}A(m/p^2)\right\}. \tag{5.5}$$

Since $A(m) = 0$ unless $m \equiv -r \mod 8$, we have that $A^*(m) = 0$ unless $m \equiv -r \mod 8$.

From the definition of $T^J(p^2)$ (see Lemma 2.6), by using the facts that $\frac{r+1}{2} \equiv k+1 \mod 2$ and $[D_r^\sharp : D_r] = 4$, we obtain

$$(\phi|_{k+\frac{1}{2},D_r}T^J(p^2))(\tau,z) \quad = \quad \sum_{n'\in\mathbb{Z}}\sum_{r'\in D_r^\sharp} C^*(n',r')e(n'\tau + \beta(r',z)),$$

where

$$\begin{aligned} C^*(n',r') \quad &= \quad C(p^2n', pr') \\ &\quad + p^{k-1}\left(\frac{(-1)^{k+1}}{p}\right)\left(\frac{8(n'-\beta(r'))}{p}\right)C(n',r') \\ &\quad + p^{2k-1}\sum_{\lambda\in D_r/pD_r} C\left(\frac{1}{p^2}(n'-\beta(r',\lambda)+\beta(\lambda)), \frac{1}{p}(r'-\lambda)\right). \end{aligned} \tag{5.6}$$

Since ϕ belongs to $J^{cusp}_{k+\frac{r+1}{2},D_r}$ and since $k + \frac{r+1}{2}$ is odd, we have $\phi(\tau,-z) = -\phi(\tau,z)$. Thus $C(n',-v_1) = -C(n',v_1)$. Now we will show $C^*(n',v_1) = A^*(8n'-r)$ for any $n' \in \mathbb{Z}$.

We have $pv_1 \equiv \left(\frac{-4}{p}\right)v_1 \mod D_r$. We remark $\beta(v_1) = r/8$.

Since $C(n',r')$ is determined by the pair $(n'-\beta(r'),\ r' \bmod D_r)$, we have

$$\begin{aligned} C(p^2n', pv_1) \quad &= \quad C\left(p^2n' + \frac{1-p^2}{8}r, \left(\frac{-4}{p}\right)v_1\right) = \left(\frac{-4}{p}\right)C\left(p^2n' + \frac{1-p^2}{8}r, v_1\right) \\ &= \quad \left(\frac{-4}{p}\right)A(p^2(8n'-r)). \end{aligned}$$

Similarly, if $\frac{1}{p}(v_1-\mu) \in D_r^\sharp$ with $\mu \in D_r$, then

$$\frac{1}{p}(v_1-\mu) \quad \equiv \quad \frac{p^2}{p}(v_1-\mu) \quad \equiv \quad pv_1 \quad \equiv \quad \left(\frac{-4}{p}\right)v_1 \mod D_r.$$

And, since there exists only one $\mu \in D_r$ modulo pD_r such that $\frac{1}{p}(v_1-\mu)\in D_r^\sharp$, we take such μ and we have

$$\begin{aligned}
&\sum_{\lambda\in D_r/pD_r} C\left(\frac{1}{p^2}(n'-\beta(v_1,\lambda)+\beta(\lambda)), \frac{1}{p}(v_1-\lambda)\right)\\
&= C\left(\frac{1}{p^2}(n'-\beta(v_1,\mu)+\beta(\mu)), \frac{1}{p}(v_1-\mu)\right) = C\left(\frac{8n'-r+p^2r}{8p^2}, \left(\frac{-4}{p}\right)v_1\right)\\
&= \left(\frac{-4}{p}\right) C\left(\frac{8n'-r+p^2r}{8p^2}, v_1\right) = \left(\frac{-4}{p}\right) A\left(\frac{1}{p^2}(8n'-r)\right).
\end{aligned}$$

Therefore the identity (5.6) for $r'=v_1$ is

$$\begin{aligned}
C^*(n',v_1) &= \left(\frac{-4}{p}\right)\Big\{A(p^2(8n'-r))\\
&\quad + p^{k-1}\left(\frac{(-1)^k}{p}\right)\left(\frac{8n'-r}{p}\right)A(8n'-r)+p^{2k-1}A\left(\frac{1}{p^2}(8n'-r)\right)\Big\}\\
&= A^*(8n'-r).
\end{aligned}$$

Since

$$\mathcal{J}^{odd}_{r,k}(\phi|T^J(p))(\tau) = \sum_{n'\in\mathbb{Z}} C^*(n',v_1)q^{(8n'-r)/8}$$

and

$$((\mathcal{J}^{odd}_{r,k}\phi)|\tilde{T}(p^2))(\tau) = \sum_{m\in\mathbb{Z}} A^*(m)q^{m/8},$$

we obtain $\mathcal{J}^{odd}_{r,k}(\phi|T^J(p)) = \mathcal{J}^{odd}_{r,k}(\phi)|\tilde{T}(p^2)$. □

Theorem 5.3 ([Ya 14, Thm. 2]). *If $r\in\{1,3,5,7\}$ and if $s\in 2\mathbb{Z}$, then*

$$\eta^{3r}M_s(1) \cong \begin{cases} S^{new,+}_{3r+2s-1}(2), & \text{if } r=1,3,\\ S^{new,-}_{3r+2s-1}(2), & \text{if } r=5,7,\end{cases}$$

as Hecke modules. Here, we use the twisted Hecke operators $\tilde{T}(p^2)$ of half-integral weight for the left-hand side (see §2.6) and use the usual Hecke operators of integral weight for the right-hand side.

Remark that the notation $S^{new}_{3r+2s-1}\big(2,-\left(\frac{8}{r}\right)\big)$ in [Ya 14, Thm.2] is equivalent to the notation $S^{new,\epsilon}_{3r+2s-1}(2)$ in this paper, where $\epsilon = -\left(\frac{8}{r}\right) i^{3r+2s-1} = \left(\frac{-8}{r}\right)$. Hence, $\epsilon=1$ or -1 according as $r=1,3$ or $r=5,7$. In [Ya 14, Thm.2] the usual Hecke operator $T(p^2)$ on $\eta^{3r}M_s(1)$ is applied, while we apply the twisted Hecke operator $\tilde{T}(p^2)$. Therefore we drop the symbol $\otimes\left(\frac{-4}{\cdot}\right)$ from $S^{new}_{3r+2s-1}\big(2,-\left(\frac{8}{r}\right)\big)\otimes\left(\frac{-4}{\cdot}\right)$ in [Ya 14, Thm.2].

Remark also that when $r=1$ and $s=0$, the right hand side $S^{new}_2(2)$ of the above isomorphism must be $M^{new}_2(2)$, and where $M^{new}_2(2) = \mathbb{C}E^{(2)}_2$ (see §7.2, the proof of

Theorem 1.4). Here $E_2^{(2)}$ has been defined in Theorem 1.4, which is a modular form of weight 2 with level 2.

By the substitution $(r,s) \to (8-r, k-12+\frac{3r+1}{2})$ into the isomorphism in Theorem 5.3, we have the following corollary.

Corollary 5.4. *If $r \in \{1,3,5,7\}$ and if $k+\frac{r+1}{2} \equiv 1 \bmod 2$ with $k \geq 2$, then*

$$\eta^{3(8-r)} M_{k-12+\frac{3r+1}{2}}(1) \quad \cong \quad \begin{cases} S_{2k}^{new,-}(2), & \text{if } r = 1,3, \\ S_{2k}^{new,+}(2), & \text{if } r = 5,7, \end{cases}$$

as Hecke modules.

Proof of Theorem 5.1. Theorem 5.1 follows from Proposition 5.2 and Corollary 5.4 (Theorem 5.3). □

6. Jacobi forms of even weight

Let the notation be as in §4. In this section we assume that $k+\frac{r+1}{2}$ is an even integer and $r = 1, 3, 5$ or 7. The purpose of this section is to show Theorem 6.1.

6.1. Jacobi forms of even weight of index D_r.

The vector space $S_{k+\frac{1}{2}}^{+,-r}(8)$ has been defined in §2.4.

Theorem 6.1. *Let $k+\frac{r+1}{2}$ be an even integer with $k \geq 0$. Then, we have*

$$J_{k+\frac{r+1}{2},D_r}^{cusp,new} \quad \cong \quad S_{k+\frac{1}{2}}^{new,+,-r}(8) \quad \cong \quad S_{2k}^{new,\epsilon_2}(2) \tag{6.1}$$

as Hecke modules, where we recall $\epsilon_2 = -\left(\frac{-8}{r}\right) = -1$ or 1 according as $r = 1, 3$ or $r = 5, 7$, and where $S_{k+\frac{1}{2}}^{new,+,-r}(8)$, which will be defined before Theorem 6.7, is the subspace of all newforms in $S_{k+\frac{1}{2}}^{+,-r}(8)$.

The first isomorphism of (6.1) will be shown in Corollary 6.8 of Proposition 6.3, while the second isomorphism has been essentially obtained by [U-Y 10, Thm. 1] (see Theorem 6.7).

6.2. Jacobi forms and modular forms of half-integral weight.

Lemma 6.2. *The group $\Gamma_0(8)$ is generated by the following two types of matrices:*

$$u(s) \quad := \quad \begin{pmatrix} 1 & 0 \\ 8s & 1 \end{pmatrix} \ (s \in \mathbb{Z}), \quad v(a) \quad := \quad \begin{pmatrix} a & 1 \\ a^2-1 & a \end{pmatrix} \ (a = \pm 1, \pm 3).$$

Proof. We denote by Γ' the group which is generated by the above two types of matrices. We have $u(s)v(a)u(s) = v(a+8s)$. Thus $v(a)$ belongs to Γ' for any odd integer a. Note that $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) = v(1)v(-1) \in \Gamma'$ and $v(a)^{-1} = -v(-a) \in \Gamma'$.

We take $A = \begin{pmatrix} x & y \\ 8z & w \end{pmatrix} \in \Gamma_0(8)$. We can take an odd integer a, such that $|ay+w| < |y|$. Since $v(a)A = \begin{pmatrix} * & ay+w \\ * & * \end{pmatrix}$, by repeating to multiply such $v(a)$ to A, we can reduce the absolute value of the right-upper part of A. Thus we can assume $y = 0$. Then we have $A = \left(\begin{smallmatrix} \pm 1 & 0 \\ 8z & \pm 1 \end{smallmatrix}\right) \in \Gamma'$. Thus $\Gamma' = \Gamma_0(8)$. □

We recall from §4 that $\phi \in J_{k+\frac{r+1}{2},D_r}$ has the decomposition:

$$\phi(\tau,z) \;=\; \sum_{j=0}^{3} h_j(\tau)\theta_{r,j}(\tau,z), \tag{6.2}$$

where

$$h_j(\tau) \;=\; \sum_{n'\in\mathbb{Z}} C(n',v_j)e((n'-\beta(v_j))\tau),$$

and where $C(n',v_j)$ is the (n',v_j)-th Fourier coefficient of ϕ (see Lemma 2.3). We define

$$(\mathcal{J}^{even}_{r,k}(\phi))(\tau) \;:=\; \sum_{j=0}^{3} h_j(8\tau). \tag{6.3}$$

Recall that the vector space $M^{+,-r}_{k+\frac{1}{2}}(8)$ has been defined in §2.4.

Proposition 6.3. *Assume that $r = 1$, 3, 5 or 7 and that $k + \frac{r+1}{2}$ is an even integer with $k \geq 0$. Then, the map $\mathcal{J}^{even}_{r,k}$ gives an isomorphism*

$$\mathcal{J}^{even}_{r,k} \;:\; J_{k+\frac{r+1}{2},D_r} \;\xrightarrow{\cong}\; M^{+,-r}_{k+\frac{1}{2}}(8)$$

as Hecke modules. It means, for any $\phi \in J_{k+\frac{r+1}{2},D_r}$, we have

$$\mathcal{J}^{even}_{r,k}(\phi|T^J(p)) \;=\; (\mathcal{J}^{even}_{r,k}(\phi))|T(p^2)$$

for any odd prime p. Moreover, the restriction of $\mathcal{J}^{even}_{r,k}$ on $J^{cusp}_{k+\frac{r+1}{2},D_r}$ gives the isomorphism

$$\mathcal{J}^{even}_{r,k} \;:\; J^{cusp}_{k+\frac{r+1}{2},D_r} \;\xrightarrow{\cong}\; S^{+,-r}_{k+\frac{1}{2}}(8)$$

This proposition follows from three facts: $\mathcal{J}^{even}_{r,k}$ is an injective map (Lemma 6.4), $\mathcal{J}^{even}_{r,k}$ is surjective (Lemma 6.5), and $\mathcal{J}^{even}_{r,k}$ is compatible with the action of Hecke operators (Lemma 6.6). We show these three lemmas.

Lemma 6.4. *Let $\phi \in J_{k+\frac{r+1}{2},D_r}$. Then $\mathcal{J}^{even}_{r,k}(\phi) \in M^{+,-r}_{k+\frac{1}{2}}(8)$. Furthermore, the map $\mathcal{J}^{even}_{r,k} : J_{k+\frac{r+1}{2},D_r} \to M^{+,-r}_{k+\frac{1}{2}}(8)$ is injective. If $\phi \in J^{cusp}_{k+\frac{r+1}{2},D_r}$, then $\mathcal{J}^{even}_{r,k}(\phi) \in S^{+,-r}_{k+\frac{1}{2}}(8)$.*

Proof. For $\phi \in J_{k+\frac{r+1}{2},D_r}$ we put $g = \mathcal{J}^{even}_{r,k}(\phi)$. Since $\beta(v_0) = 0$, $\beta(v_2) = 1/2$ and $\beta(v_1) = \beta(v_3) = r/8$, and since $h_1 = h_3$, the injectivity of $\mathcal{J}^{even}_{r,k}$ follows from the Fourier expansion of g.

We show $g \in M^{+,-r}_{k+\frac{1}{2}}(8)$. By virtue of Lemma 6.2, it is sufficient to prove the transformation formula of g for the generators $u(s)$ and $v(a)$ of $\Gamma_0(8)$. To show this, we show the transformation formula of $g(\tau/8) = \sum_{j=0}^{3} h_j(\tau)$ for

$$u'(s) \ := \ \begin{pmatrix} 8 & 0 \\ 0 & 1 \end{pmatrix} u(s) \begin{pmatrix} 8 & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \quad (s \in \mathbb{Z})$$

and for

$$v'(a) \ := \ \begin{pmatrix} 8 & 0 \\ 0 & 1 \end{pmatrix} v(a) \begin{pmatrix} 8 & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a & 8 \\ (a^2-1)/8 & a \end{pmatrix} \quad (a = \pm 1, \pm 3).$$

Remark the identity $u'(s) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. By virtue of the identities (4.2), (6.2), and by using the transformation formula of ϕ, we have

$$h_j(-\tau^{-1}) \ = \ \frac{1}{2}\tau^{k+\frac{1}{2}} e(r/8) \sum_{t=0}^{3} e(jtr/4) h_{r_t}(\tau).$$

For any $s \in \mathbb{Z}$, we obtain

$$\begin{aligned} h_j(-(\tau - s)^{-1}) \ &= \ \frac{1}{2}(\tau - s)^{k+\frac{1}{2}} e(r/8) \sum_{t=0}^{3} e(jtr/4) h_t(\tau - s) \\ &= \ \frac{1}{2}(\tau - s)^{k+\frac{1}{2}} e(r/8) \sum_{t=0}^{3} e(jtr/4) e(t^2 rs/8) h_t(\tau). \end{aligned}$$

Hence, by virtue of the identity (4.2), we have

$$\begin{aligned} h_j(-(-\tau^{-1} - s)^{-1}) \ &= \ \frac{1}{2}\left(-\tau^{-1} - s\right)^{k+\frac{1}{2}} e(r/8) \sum_{t=0}^{3} e(jtr/4) e(t^2 rs/8) h_t(-\tau^{-1}) \\ &= \ \frac{1}{4}\left(-\tau^{-1} - s\right)^{k+\frac{1}{2}} \tau^{k+\frac{1}{2}} e(r/4) \sum_{t=0}^{3} e(jtr/4) e(t^2 rs/8) \\ &\quad \times \sum_{u=0}^{3} e(tur/4) h_u(\tau). \end{aligned}$$

Since $(-\tau^{-1} - s)^{\frac{1}{2}} \tau^{\frac{1}{2}} = (s\tau + 1)^{\frac{1}{2}} e(1/4)$ and since $2k + 1 \equiv -r \mod 4$, we have $(-\tau^{-1} - s)^{k+\frac{1}{2}} \tau^{k+\frac{1}{2}} = (s\tau + 1)^{k+\frac{1}{2}} e(-r/4)$. Since $\sum_{j=0}^{3} e(jtr/4) = 4\delta_{t,0}$, where $\delta_{t,0}$

denotes the Kronecker delta, we have

$$\begin{aligned}\sum_{j=0}^{3} h_j(\tau(s\tau+1)^{-1}) &= \sum_{j=0}^{3} h_j(-(-\tau^{-1}-s)^{-1}) \\ &= (s\tau+1)^{k+\frac{1}{2}} \sum_{u=0}^{3} h_u(\tau).\end{aligned}$$

Therefore $g\left(\dfrac{\tau}{8(s\tau+1)}\right) = (s\tau+1)^{k+\frac{1}{2}} g\left(\dfrac{\tau}{8}\right)$ and we obtain $g|_{k+\frac{1}{2}} u(s) = g$ for any $s \in \mathbb{Z}$.

For $a \in \{\pm 1\}$, we have $v'(a) = \begin{pmatrix} a & 8 \\ 0 & a \end{pmatrix}$. Thus $\sum_{j=0}^{3} h_j(\tau + 8a^{-1}) = \sum_{j=0}^{3} h_j(\tau)$ for $v'(a)$ $(a \in \{\pm 1\})$. Therefore $g|_{k+\frac{1}{2}} v(a) = g$ for $a \in \{\pm 1\}$.

For $a \in \{\pm 3\}$, we obtain $v'(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$. We have

$$\sum_{j=0}^{3} h_j(\tau + a) = \sum_{j=0}^{3} e(-aj^2 r/8) h_j(\tau)$$

and

$$\begin{aligned}\sum_{j=0}^{3} h_j(-\tau^{-1} + a) &= \sum_{j=0}^{3} e(-aj^2 r/8) h_j(-\tau^{-1}) \\ &= \frac{1}{2} \tau^{k+\frac{1}{2}} e(r/8) \sum_{j=0}^{3} e(-aj^2 r/8) \sum_{t} e(jtr/4) h_t(\tau).\end{aligned}$$

Thus

$$\begin{aligned}&\sum_{j=0}^{3} h_j\left(-\frac{1}{\tau + a} + a\right) \\ &= \frac{1}{2} (\tau + a)^{k+\frac{1}{2}} e\left(\frac{r}{8}\right) \sum_{j=0}^{3} e\left(\frac{-aj^2 r}{8}\right) \sum_{t=0}^{3} e\left(\frac{jtr}{4}\right) h_t(\tau + a) \\ &= \frac{1}{2} (\tau + a)^{k+\frac{1}{2}} e\left(\frac{r}{8}\right) \sum_{j=0}^{3} e\left(\frac{-aj^2 r}{8}\right) \sum_{t=0}^{3} e\left(\frac{jtr}{4}\right) e\left(\frac{-at^2 r}{8}\right) h_t(\tau).\end{aligned}$$

It is straight-forward to check that

$$\sum_{j=0}^{3} e\left(\frac{-aj^2 r}{8}\right) e\left(\frac{jtr}{4}\right) e\left(\frac{-at^2 r}{8}\right) = 2\, e\left(\frac{-ar}{8}\right).$$

Since $(a-1)(2k+1+r) \equiv 0 \mod 8$, we have

$$\begin{aligned}\sum_{j=0}^{3} h_j((a\tau+(a^2-1))(\tau+a)^{-1}) &= \sum_{j=0}^{3} h_j(-(\tau+a)^{-1}+a)\\ &= (\tau+a)^{k+\frac{1}{2}}\, e((1-a)r/8) \sum_{t=0}^{3} h_t(\tau)\\ &= (\tau+a)^{k+\frac{1}{2}}\, e((a-1)(2k+1)/8) \sum_{t=0}^{3} h_t(\tau).\end{aligned}$$

Thus

$$g\left(\frac{a\tau+(a^2-1)}{8(\tau+a)}\right) = (\tau+a)^{k+\frac{1}{2}}\, e((a-1)(2k+1)/8) g\left(\frac{\tau}{8}\right).$$

Since $j(v(a),\tau) = (8\tau+a)^{\frac{1}{2}} e((a-1)/8)$ for $a \in \{\pm 3\}$, we have $g|_{k+\frac{1}{2}} v(a) = g$.

Therefore $g \in M_{k+\frac{1}{2}}(8)$. Since $\beta(v_0)=0$, $\beta(v_2)=1/2$ and $\beta(v_1)=\beta(v_3)=r/8$, and by the construction of g, we obtain $g \in M^{+,-r}_{k+\frac{1}{2}}(8)$.

Now, we assume that ϕ is a Jacobi cusp form. Since the constant term of ϕ is zero, the constant term of h_0 is zero. Thus, by using transformation formula of $\{h_j\}_j$, we have the fact that $|\mathrm{Im}(\tau)^{-k/2-1/4} g(\tau)|$ is bounded on $\mathfrak{H}$. Hence g is a cusp form and $g \in S^{+,-r}_{k+\frac{1}{2}}(8)$. □

Lemma 6.5. *Let $g \in M^{+,-r}_{k+\frac{1}{2}}(8)$. We write*

$$g(\tau) = \sum_{\substack{n\in\mathbb{Z}\\ n\equiv 0,4,-r \mod 8}} c_g(n) q^n = \sum_{m=0,4,-r} f_m(8\tau),$$

where $f_m(\tau) := \sum_{\substack{n\in\mathbb{Z}\\ n\equiv m \mod 8}} c_g(n) q^{n/8}$. We put $(J(g))(\tau,z) := \sum_{j=0}^{3} h_j(\tau)\theta_{r,j}(\tau,z)$, where we set

$$h_j(\tau) := \begin{cases} f_{2j}(\tau), & \text{if } j = 0,2,\\ \frac{1}{2} f_{-r}(\tau), & \text{if } j=1,3.\end{cases}$$

Then, $J(g)$ belongs to $J_{k+\frac{r+1}{2},D_r}$. Moreover, the map J is the inverse map of $\mathcal{J}^{even}_{r,k}$. If $g \in S^{+,-r}_{k+\frac{1}{2}}(8)$, then $J(g) \in J^{cusp}_{k+\frac{r+1}{2},D_r}$.

Proof. For simplicity, we write $\phi = J(g)$. The transformation equations in Definition 2.1(ii) and (iii) for ϕ follow directly from the definition of ϕ. Thus it is sufficient to show the transformation equation in Definition 2.1 (i) for ϕ. The formula

$\phi(\tau+1,z)=\phi(\tau,z)$ follows from the definition of ϕ and from the transformation equation of $\theta_{r,j}$ with respect to $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. Therefore it is sufficient to show the transformation equation of ϕ with respect to $\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$:

$$\phi(-\tau^{-1},\tau^{-1}z) = \tau^{k+\frac{r+1}{2}}e(\tau^{-1}\beta(z))\phi(\tau,z). \tag{6.4}$$

To show (6.4), we show

$$f_0(-\tau^{-1}) = \frac{1}{2}\sqrt{\tau}^{2k+1}e(r/8)\left\{f_0(\tau)+f_4(\tau)+f_{-r}(\tau)\right\}, \tag{6.5}$$

$$f_4(-\tau^{-1}) = \frac{1}{2}\sqrt{\tau}^{2k+1}e(r/8)\left\{f_0(\tau)+f_4(\tau)-f_{-r}(\tau)\right\}, \tag{6.6}$$

$$f_{-r}(-\tau^{-1}) = \sqrt{\tau}^{2k+1}e(r/8)\left\{f_0(\tau)-f_4(\tau)\right\}. \tag{6.7}$$

Since

$$\frac{1}{4}\sum_{\substack{a \bmod 8\\(a,2)=2}} g\left(\frac{\tau+a}{8}\right) = f_0(\tau)+f_4(\tau),$$

we have

$$\begin{aligned} f_0(-\tau^{-1}) &= \frac{1}{8}\sum_{a \bmod 8} g\left(\frac{-\tau^{-1}+a}{8}\right) \\ &= \frac{1}{2}\left(f_0(-\tau^{-1})+f_4(-\tau^{-1})\right)+\frac{1}{8}\sum_{\substack{a \bmod 8\\(a,2)=1}} g\left(\frac{a\tau-1}{8\tau}\right). \end{aligned}$$

Thus

$$\begin{aligned} f_0(-\tau^{-1})-f_4(-\tau^{-1}) &= \frac{1}{4}\sum_{\substack{a \bmod 8\\(a,2)=1}} g\left(\frac{a\tau-1}{8\tau}\right) \\ &= \frac{1}{4}\sum_{a=\pm1,\pm3} g\left(\begin{pmatrix}a & (a^2-1)/8\\ 8 & a\end{pmatrix}\begin{pmatrix}1 & -a\\ 0 & 8\end{pmatrix}\cdot\tau\right) \\ &= \frac{1}{4}\sum_{a=\pm1,\pm3} j\left(\begin{pmatrix}a & (a^2-1)/8\\ 8 & a\end{pmatrix},\begin{pmatrix}1 & -a\\ 0 & 8\end{pmatrix}\cdot\tau\right)^{2k+1} g\left(\frac{\tau-a}{8}\right). \end{aligned}$$

Now, by using the formula

$$j\left(\begin{pmatrix}a & (a^2-1)/8\\ 8 & a\end{pmatrix},\begin{pmatrix}1 & -a\\ 0 & 8\end{pmatrix}\tau\right)=\sqrt{\left(\frac{-4}{a}\right)}^{-1}\left(\frac{8}{a}\right)\sqrt{\tau}=e\left(\frac{a-1}{8}\right)\sqrt{\tau}$$

for any odd integer a, we have

$$\begin{aligned} f_0(-\tau^{-1}) - f_4(-\tau^{-1}) &= \frac{1}{4}\sqrt{\tau}^{2k+1} \sum_{a=\pm 1, \pm 3} e\left(\frac{(a-1)(2k+1)}{8}\right) g\left(\frac{\tau - a}{8}\right) \\ &= \frac{1}{4}\sqrt{\tau}^{2k+1} \sum_{r_i = 0,4,-r} \sum_{a=\pm 1,\pm 3} e\left(\frac{(a-1)(2k+1) - a r_i}{8}\right) f_{r_i}(\tau). \end{aligned}$$

From this identity, since $2k+1+r \equiv 0 \mod 4$, we obtain

$$f_0(-\tau^{-1}) - f_4(-\tau^{-1}) = \sqrt{\tau}^{2k+1} e(r/8) f_{-r}(\tau). \tag{6.8}$$

Since $\sqrt{-\tau^{-1}}^{-1} = -i\sqrt{\tau}$, the substitution $\tau \to -\tau^{-1}$ into (6.8) gives the identity (6.7).

Since $-\frac{1}{\tau+1} = -1 + \frac{\tau}{\tau+1}$, the substitution $\tau \to \tau + 1$ into (6.8) gives

$$f_0\left(\frac{\tau}{\tau+1}\right) + f_4\left(\frac{\tau}{\tau+1}\right) = \sqrt{\tau+1}^{2k+1} f_{-r}(\tau). \tag{6.9}$$

Since $-\frac{1}{-\tau^{-1}-1} = \frac{\tau}{\tau+1}$, the substitution $\tau \to -\tau^{-1} - 1$ into (6.8) gives

$$f_0\left(\frac{\tau}{\tau+1}\right) - f_4\left(\frac{\tau}{\tau+1}\right) = \sqrt{-\tau^{-1}-1}^{2k+1} e\left(\frac{1}{4}r\right) f_{-r}(-\tau^{-1}). \tag{6.10}$$

Hence, by adding both sides of (6.9) and (6.10), we obtain

$$f_0\left(\frac{\tau}{\tau+1}\right) = \frac{1}{2}\left\{\sqrt{\tau+1}^{2k+1} f_{-r}(\tau) + \sqrt{-\tau^{-1}-1}^{2k+1} e\left(\frac{1}{4}r\right) f_{-r}(-\tau^{-1})\right\}.$$

Since $\sqrt{-\tau^{-1}-1}\sqrt{\tau} = e(1/4)\sqrt{\tau+1}$, we have

$$f_0\left(\frac{\tau}{\tau+1}\right) = \frac{1}{2}\sqrt{\tau+1}^{2k+1}\left\{f_{-r}(\tau) + \sqrt{\tau}^{-2k-1} e\left(\frac{1}{4}(2k+1+r)\right) f_{-r}(-\tau^{-1})\right\}.$$

By virtue of (6.7) and since $2k+1+r \equiv 0 \mod 4$, we obtain

$$f_0\left(\frac{\tau}{\tau+1}\right) = \frac{1}{2}\sqrt{\tau+1}^{2k+1}\left\{f_{-r}(\tau) + e\left(\frac{r}{8}\right)(f_0(\tau) - f_4(\tau))\right\}.$$

The substitution $\tau \to \tau - 1$ into this identity gives the identity (6.5). The identities (6.5) and (6.8) provide the identity (6.6).

The identity

$$\begin{pmatrix} h_0(-\tau^{-1}) \\ h_2(-\tau^{-1}) \\ h_1(-\tau^{-1}) \\ h_3(-\tau^{-1}) \end{pmatrix} = \frac{1}{2}\sqrt{\tau}^{2k+1} e(r/8) \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & e(r/4) & -e(r/4) \\ 1 & -1 & -e(r/4) & e(r/4) \end{pmatrix} \begin{pmatrix} h_0(\tau) \\ h_2(\tau) \\ h_1(\tau) \\ h_3(\tau) \end{pmatrix}$$

follows from the identities (6.5), (6.6) and (6.7) and from the fact $h_r = h_{-r}$.

This identity and the transformation law of $\{\theta_{r,j}\}$ (see (2.1)) provide the identity (6.4). Thus the transformation law in Definition 2.1 (i) for ϕ follows. We therefore have $J(g) = \phi \in J_{k+\frac{r+1}{2},D_r}$.

If $g \in S^{+,-r}_{k+\frac{1}{2}}(8)$, then the constant term of f_0 is zero. Thus the constant term of h_0 is also zero. Hence ϕ is a Jacobi cusp form (see Lemma 4.1) and $J(g) \in J^{cusp}_{k+\frac{r+1}{2},D_r}$.

By the construction of J, it is clear that J is the inverse map of $\mathcal{J}^{even}_{r,k}$. □

Lemma 6.6. *The map $\mathcal{J}^{even}_{r,k}$ is compatible with the action of Hecke operators:*

$$\mathcal{J}^{even}_{r,k}(\phi|T^J(p)) \;=\; (\mathcal{J}^{even}_{r,k}(\phi))|T(p^2)$$

for any $\phi \in J_{k+\frac{r+1}{2},D_r}$ and for any odd prime p.

Proof. The proof of this lemma is similar to the case of $\mathcal{J}^{odd}_{k,r}$ (see the proof of Proposition 5.2). We show the identity between the Fourier coefficients of $\mathcal{J}^{even}_{r,k}(\phi|T^J(p))$ and those of $(\mathcal{J}^{even}_{r,k}(\phi))|T(p^2)$.

We take the Fourier expansion of $\phi \in J_{k+\frac{r+1}{2},D_r}$:

$$\phi(\tau,z) \;=\; \sum_{n'\in\mathbb{Z}} \sum_{r'\in D_r^\sharp} C(n',r')e(n'\tau + \beta(r',z)).$$

Since $k+\frac{r+1}{2}$ is even, we have $\phi(\tau,-z) = \phi(\tau,z)$. Therefore $C(n',-r') = C(n',r')$ holds for any $(n',r') \in \mathbb{Z}\times D_r^\sharp$.

From the definition of $T^J(p^2)$ (see Lemma 2.6), we obtain

$$(\phi|_{k+\frac{1}{2},D_r}T^J(p^2))(\tau,z) \;=\; \sum_{n'\in\mathbb{Z}} \sum_{r'\in D_r^\sharp} C^{**}(n',r')e(n'\tau + \beta(r',z)),$$

where

$$\begin{aligned} C^{**}(n',r') \;=\;& C(p^2n',pr') \\ &+p^{k-1}\left(\frac{(-1)^k}{p}\right)\left(\frac{8(n'-\beta(r'))}{p}\right)C(n',r') \\ &+p^{2k-1}\sum_{\lambda\in D_r/pD_r} C\left(\frac{1}{p^2}(n'-\beta(r',\lambda)+\beta(\lambda)),\frac{1}{p}(r'-\lambda)\right). \end{aligned} \tag{6.11}$$

Remark that the second term of the right-hand side is different from the one in (5.6).

For simplicity, we write $g = \mathcal{J}^{even}_{r,k}(\phi)$. Then g has a Fourier expansion of the form $g(\tau) = \sum_{m=0}^{\infty} A(m)q^m$. The Fourier expansion of $g|_{k+\frac{1}{2}}T(p^2)$ is

$$(g|_{k+\frac{1}{2}}T(p^2))(\tau) \;=\; \sum_{m=0}^{\infty} A^{**}(m)q^m,$$

where

$$A^{**}(m) \;=\; A(p^2m) + \left(\frac{(-1)^k m}{p}\right) p^{k-1}A(m) + p^{2k-1}A(m/p^2). \tag{6.12}$$

By the construction of $\mathcal{J}^{even}_{r,k}$, we have

$$C(n',r') \;=\; \begin{cases} A(8(n'-\beta(r'))), & \text{if } r' \equiv v_0, v_2 \mod D_r, \\ \frac{1}{2}A(8(n'-\beta(r'))), & \text{if } r' \equiv v_1, v_3 \mod D_r, \end{cases}$$

for any $(n',r') \in \mathbb{Z} \times D^{\sharp}_r$. Remark that $C(n',v_1) = C(n',v_3)$.

By comparing the right-hand sides of (6.11) and of (6.12), we obtain

$$C^{**}(n',v_j) \;=\; \begin{cases} A^{**}(8(n'-\beta(v_j))), & \text{if } r' = v_0, v_2, \\ \frac{1}{2}A^{**}(8(n'-\beta(v_j))), & \text{if } r' = v_1, v_3. \end{cases}$$

We therefore conclude $\mathcal{J}^{even}_{r,k}(\phi|T^J(p)) = (\mathcal{J}^{even}_{r,k}(\phi))|T(p^2)$. □

Proof of Proposition 6.3. Proposition 6.3 follows from Lemma 6.4 (injectivity of $\mathcal{J}^{even}_{r,k}$), Lemma 6.5 (surjectivity of $\mathcal{J}^{even}_{r,k}$) and Lemma 6.6 (compatibility of $\mathcal{J}^{even}_{r,k}$ with Hecke operators). □

6.3. **Shimura correspondence and newforms.** The newforms in $S^+_{k+\frac{1}{2}}(8)$ has been introduced by [U-Y 10, Introduction]. We recall the definition of operators $U(d)$ and $U_k(4) = U(4)\wp_k$:

$$\Big(\sum_{n\in\mathbb{Z}} a_n q^n\Big)|U(d) \;=\; \sum_{n\in\mathbb{Z}} a_{dn}q^n, \qquad \Big(\sum_{n\in\mathbb{Z}} a_n q^n\Big)|\wp_k \;=\; \sum_{\substack{n\in\mathbb{Z} \\ (-1)^k n \equiv 0,1 \mod 4}} a_n q^n.$$

for any formal power q-series $\sum_{n\in\mathbb{Z}} a_n q^n$. It is shown in [U-Y 10, Thm. 1 (1), Prop. 6] that $S^+_{k+\frac{1}{2}}(4)|U_k(4) := \left\{ f|U_k(4) \;:\; f \in S^+_{k+\frac{1}{2}}(4)\right\}$ is a subspace of $S^+_{k+\frac{1}{2}}(8)$. We define

$$S^{old,+}_{k+\frac{1}{2}}(8) \;:=\; S^+_{k+\frac{1}{2}}(4) + S^+_{k+\frac{1}{2}}(4)|U_k(4).$$

The space of newforms $S^{new,+}_{k+\frac{1}{2}}(8)$ is defined by the orthogonal complement of $S^{old,+}_{k+\frac{1}{2}}(8)$ in $S^+_{k+\frac{1}{2}}(8)$. It is known in [U-Y 10, Thm. 1 (1)], that

$$\begin{aligned} S^{old,+}_{k+\frac{1}{2}}(8) &= S^+_{k+\frac{1}{2}}(4) \oplus S^+_{k+\frac{1}{2}}(4)|U_k(4), \\ S^+_{k+\frac{1}{2}}(8) &= S^{new,+}_{k+\frac{1}{2}}(8) \oplus S^{old,+}_{k+\frac{1}{2}}(8). \end{aligned}$$

We put

$$S^{new,+,-r}_{k+\frac{1}{2}}(8) \;:=\; S^{new,+}_{k+\frac{1}{2}}(8) \cap S^{+,-r}_{k+\frac{1}{2}}(8).$$

The following theorem essentially follows from [U-Y 10, Thm. 1, Prop. 4, Cor. 2].

Theorem 6.7 ([U-Y 10])**.** *For any $k \in \mathbb{Z}$ and for any $r \in \{1,3,5,7\}$, we have*

$$S^{new,+,-r}_{k+\frac{1}{2}}(8) \;\cong\; S^{new,\epsilon_2}_{2k}(2)$$

as Hecke module, where we recall $\epsilon_2 = -\left(\frac{-8}{r}\right)$.

Proof. By virtue of [U-Y 10, Thm. 1 (3)], there exists an isomorphism map

$$J_k \;:\; S^{new,+}_{k+\frac{1}{2}}(8) \to S^{new}_{2k}(2)$$

as Hecke module. That is,

$$J_k(g|T(p^2)) = J_k(g)|T(p), \quad J_k(g|U_k(4)) = J_k(g)|U(2)$$

for any $g \in S^{new,+}_{k+\frac{1}{2}}(8)$ and for any odd prime p, and where $T(p)$ is the usual Hecke operator acting on $S_{2k}(2)$.

Due to [U-Y 10, Cor. 2, Prop. 4], it is known that the identity

$$2^{1-k} g|U_k(4) \;=\; -\epsilon g$$

is equivalent to the condition

$$a_g(n) = 0 \text{ if } \left(\frac{8}{(-1)^k n}\right) = -\epsilon,$$

where $a_g(n)$ is the n-th Fourier coefficient of g.

We assume $g \in S^{new,+,-r}_{k+\frac{1}{2}}(8)$ and assume that g is a Hecke eigenform. We need to show $J_k(g) \in S^{new,\epsilon_2}_{2k}(2)$. Since $g \in S^{new,+,-r}_{k+\frac{1}{2}}(8)$, we obtain $g|U_k(4) = 2^{k-1}\left(\frac{8}{(-1)^k m}\right) g$, where

$$m \;=\; \begin{cases} -5, & \text{if } r = 1, \\ -7, & \text{if } r = 3, \\ -1, & \text{if } r = 5, \\ -3, & \text{if } r = 7. \end{cases}$$

We recall that $k + \frac{r+1}{2}$ is even. Hence, $(-1)^k = (-1)^{(r+1)/2}$ and $g|U_k(4) = -\left(\frac{8}{r}\right) 2^{k-1} g$.

We put $f = J_k(g)$ and take the Fourier expansion $f = \sum a_n q^n$. We have $f|U(2) = a_2 f$. Since $J_k(g|U_k(4)) = J_k(g)|U(2)$, we have $a_2 = -\left(\frac{8}{r}\right) 2^{k-1}$. It is known that

$$f|_{2k}\left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right) \;=\; cf$$

with $c = -2^{1-k} a_2$ (see [Zh 13, Introduction], for example). Thus

$$c \;=\; \left(\frac{8}{r}\right) \;=\; (-1)^k (-1)^{\frac{r+1}{2}} \left(\frac{8}{r}\right) \;=\; -(-1)^k \left(\frac{-8}{r}\right) \;=\; \epsilon_2 i^{-2k}.$$

We therefore conclude $J_k(g) = f \in S^{new,\epsilon_2}_{2k}(2)$. □

The following corollary follows from Proposition 6.3 and Theorem 6.7. Recall the conditions $r \in \{1,3,5,7\}$ and $k \geq 0$.

Corollary 6.8. *We have*

$$J^{cusp,new}_{k+\frac{r+1}{2},D_r} \cong S^{new,+,-r}_{k+\frac12}(8).$$

as Hecke modules.

Proof. We put

$$S^{(1),-r}_{k+\frac12} := \mathcal{J}^{even}_{r,k}(J^{cusp,new}_{k+\frac{r+1}{2},D_r}), \quad S^{(0),-r}_{k+\frac12} := \mathcal{J}^{even}_{r,k}(J^{cusp,old}_{k+\frac{r+1}{2},D_r}).$$

Then, by virtue of Proposition 6.3, we have $S^{+,-r}_{k+\frac12}(8) = S^{(1),-r}_{k+\frac12} \oplus S^{(0),-r}_{k+\frac12}$. It is sufficient to show $S^{(1),-r}_{k+\frac12} = S^{new,+,-r}_{k+\frac12}(8)$.

We put $S^{old,+,-r}_{k+\frac12}(8) := S^{old,+}_{k+\frac12}(8) \cap S^{+,-r}_{k+\frac12}(8)$.

Since $S^{+}_{k+\frac12}(8) = S^{new,+}_{k+\frac12}(8) \oplus S^{old,+}_{k+\frac12}(8)$, we have $S^{+,-r}_{k+\frac12}(8) = S^{new,+,-r}_{k+\frac12}(8) \oplus S^{old,+,-r}_{k+\frac12}(8)$. Due to Theorem 6.7 and due to the strong multiplicity one theorem (see [Mi 89, Coro. 4.6.30]), we obtain $S^{(0),-r}_{k+\frac12} \subset S^{old,+,-r}_{k+\frac12}(8)$. Namely, if $\phi \in J^{cusp,old}_{k+\frac{r+1}{2},D_r}$ is a Hecke eigenform, then there exists a Hecke eigenform $f_1 \in S^{\epsilon_1}_{2k}(1)$ such that the eigenvalues of ϕ are those of f_1. If $\mathcal{J}^{even}_{r,k}(\phi) \in S^{new,+,-r}_{k+\frac12}(8)$, then there exists $f_2 \in S^{new,\epsilon_2}_{2k}(2)$ such that the eigenvalues of ϕ are those of f_2. But, then the eigenvalues of f_1 and f_2 coincide, and it is contradiction.

It is known in [U-Y 10, Thm. 1 (1)] that

$$\dim S^{+}_{k+\frac12}(8) = \dim S^{new,+}_{k+\frac12}(8) + 2\dim S^{+}_{k+\frac12}(4) = \dim S^{new,+}_{k+\frac12}(8) + 2\dim S_{2k}(1),$$

where the last identity follows from [Ko 80]. Due to (2.2) we have

$$\dim S^{+}_{k+\frac12}(8) = \begin{cases} \dim S^{+,-1}_{k+\frac12}(8) + \dim S^{+,-5}_{k+\frac12}(8), & \text{if } k \text{ is odd,} \\ \dim S^{+,-3}_{k+\frac12}(8) + \dim S^{+,-7}_{k+\frac12}(8), & \text{if } k \text{ is even.} \end{cases}$$

First, we treat the cases $r = 1$ and $r = 5$. We have

$$\begin{aligned} \dim S^{(1),-1}_{k+\frac12} + \dim S^{(0),-1}_{k+\frac12} + \dim S^{(1),-5}_{k+\frac12} + \dim S^{(0),-5}_{k+\frac12} &= \dim S^{+}_{k+\frac12}(8) \\ &= \dim S^{new,+}_{k+\frac12}(8) + 2\dim S_{2k}(1). \end{aligned}$$

By virtue of Proposition 3.4, we have $\dim S^{(0),-1}_{k+\frac12} = \dim S^{(0),-5}_{k+\frac12} = \dim S_{2k}(1)$.

Since $S^{(0),-1}_{k+\frac12} \oplus S^{(0),-5}_{k+\frac12} \subset S^{old,+,-1}_{k+\frac12}(8) \oplus S^{old,+,-5}_{k+\frac12}(8) = S^{old,+}_{k+\frac12}(8)$, and since $\dim S^{old,+}_{k+\frac12}(8) = 2\dim S^{+}_{k+\frac12}(4) = 2\dim S_{2k}(1)$, we have $S^{(0),-1}_{k+\frac12} = S^{old,+,-1}_{k+\frac12}(8)$ and $S^{(0),-5}_{k+\frac12} = S^{old,+,-5}_{k+\frac12}(8)$. Therefore $S^{(1),-1}_{k+\frac12} = S^{new,+,-1}_{k+\frac12}(8)$ and $S^{(1),-5}_{k+\frac12} = S^{new,+,-5}_{k+\frac12}(8)$.

The cases $r = 3$ and $r = 7$ follows from the same argument. □

Proof of Theorem 6.1. Theorem 6.1 follows from Corollary 6.8 (isomorphism $J^{cusp,new}_{k+\frac{r+1}{2},D_r} \cong S^{new,+,-r}_{k+\frac{1}{2}}(8)$) and Theorem 6.7 (isomorphism $S^{new,+,-r}_{k+\frac{1}{2}}(8) \cong S^{new,\epsilon_2}_{2k}(2)$). □

7. Proof of Theorems 1.1 and 1.4

7.1. Proof of Theorem 1.1.

Proof of Theorem 1.3. Theorem 1.3 follows from Theorem 5.1 (isomorphisms

$$J^{cusp}_{k+\frac{r+1}{2},D_r} \cong \eta^{24-3r} M_{k-12+\frac{3r+1}{2}}(1) \cong S^{new,\epsilon_2}_{2k}(2)$$

for $k+\frac{r+1}{2} \equiv 1 \bmod 2$) and from Theorem 6.1 (isomorphisms

$$J^{cusp,new}_{k+\frac{r+1}{2},D_r} \cong S^{new,+,-r}_{k+\frac{1}{2}}(8) \cong S^{new,\epsilon_2}_{2k}(2)$$

for $k+\frac{r+1}{2} \equiv 0 \bmod 2$). □

Remark that if $k+\frac{r+1}{2} \equiv 1 \bmod 2$, then it follows from Theorem 1.2 that $J^{old}_{k+\frac{r+1}{2},D_r} = \{0\}$ and $J^{cusp,new}_{k+\frac{r+1}{2},D_r} = J^{cusp}_{k+\frac{r+1}{2},D_r}$.

Proof of Theorem 1.1. Theorem 1.1 follows from Theorem 1.2

$$\text{(isomorphism } J_{k+\frac{r+1}{2},D_r} = J^{new}_{k+\frac{r+1}{2},D_r} \oplus J^{old}_{k+\frac{r+1}{2},D_r} \cong J^{new}_{k+\frac{r+1}{2},D_r} \oplus M^{\epsilon_1}_{2k}(1))$$

and from Theorem 1.3 (isomorphism $J^{new}_{k+\frac{r+1}{2},D_r} \cong S^{\epsilon_2}_{2k}(2)$). □

7.2. Proof of Theorem 1.4.

Proof of Theorem 1.4. We first show the identities.

(7.1) $$\dim J_{2,D_1} = \dim J_{3,D_3} = \dim M^{+,-1}_{3/2}(8) = \dim \eta^{15} M_{-6}(1) = \dim M^-_2(2) = 0.$$

The identities $\dim J_{2,D_1} = \dim J_{3,D_3} = 0$ have been obtained by [B-S 23] (see [Mo 19b, Thm. 3.29]). Due to Proposition 6.3, we have $\dim M^{+,-1}_{3/2}(8) = \dim J_{2,D_1} = 0$. The identities $\dim \eta^{15} M_{-6}(1) = \dim M_{-6}(1) = 0$ are obvious. One can check that $\dim M_2(2) = \dim M^+_2(2) = 1$ and $\dim M^-_2(2) = 0$. Thus we have the identities (7.1).

To show the rest of the claims in Theorem 1.4, we calculate the Hecke eigenvalues of $\eta^3 \in \eta^3 M_0(1)$ and those of $E_{4,D_5} \in J_{4,D_5}$.

It is well-known that η^3 has the Fourier expansion $\eta^3(\tau) = \sum_{n=1}^{\infty} \left(\frac{-4}{n}\right) n q^{n^2/8}$. Therefore, for any odd prime p, we have

$$\eta^3 | \tilde{T}(p^2) \quad = \quad (1+p)\eta^3$$

after a straight-forward calculation, where the twisted Hecke operator $\tilde{T}(p^2)$ is defined in §2.6. The Eisenstein series $E^{(2)}_2$ of weight 2 of level 2 has the Hecke eigenvalue $1+p$ for any odd prime p. Thus $\mathbb{C}\eta^3 \cong \mathbb{C}E^{(2)}_2$ as Hecke modules.

The identity $\dim J_{4,D_5} = 1$ has been obtained by [B-S 23] (see [Mo 19b, Thm. 3.29]). The form E_{4,D_5} belongs to J_{4,D_5}, and hence E_{4,D_5} is a Hecke eigenform. The constant term of E_{4,D_5} is not zero, thus the eigenvalue of E_{4,D_5} is obtained by computing the constant term of $E_{4,D_5}|T^J(p)$ (see §2.6, for the definition of $T^J(p)$). Hence, it is straightforward to check $E_{4,D_5}|T^J(p) = (1+p)E_{4,D_5}$ for any odd prime p. Thus $\mathbb{C}E_{4,D_5} \cong \mathbb{C}E_2^{(2)}$ as Hecke modules.

Due to Proposition 5.2, we obtain the identities

$$\dim J_{5,D_7} = \dim \eta^3 M_0(1) = \dim M_2^+(2) = 1$$

and

$$\mathbb{C}\psi_{5,D_7} \cong \mathbb{C}\eta^3 \cong \mathbb{C}E_2^{(2)}$$

as Hecke modules, where $\psi_{5,D_7} \in J_{5,D_7}$.

The identity $\dim J_{4,D_5} = 1$ has been obtained by [B-S 23] (see [Mo 19b, Thm. 3.29]). Due to Proposition 6.3, we obtain the isomorphism $M_{3/2}^{+,-5}(8) \cong J_{4,D_5}$. Since $\dim J_{4,D_5}^{old} = \dim M_2^-(1) = 0$, we have $J_{4,D_5} = J_{4,D_5}^{new}$. Thus $M_{3/2}^{+,-5}(8) \cong J_{4,D_5}^{new}$. We recall $E_{3/2}^{(8)} \in M_{3/2}^{+,-5}(8)$ and $E_{4,D_5} \in J_{4,D_5}^{new}$. We have $\mathbb{C}E_{3/2}^{(8)} \cong \mathbb{C}E_{4,D_5}$.

We therefore conclude

$$\dim J_{4,D_5}^{new} = \dim J_{5,D_7}^{cusp,new} = \dim M_{3/2}^{+,-5}(8) = \dim \eta^3 M_0(1) = \dim M_2^+(2) = 1$$

and

$$\mathbb{C}E_{4,D_5} \cong \mathbb{C}\psi_{5,D_7} \cong \mathbb{C}E_{3/2}^{(8)} \cong \mathbb{C}\eta^3 \cong \mathbb{C}E_2^{(2)}$$

as Hecke modules. □

8. Fourier coefficients of Jacobi-Eisenstein series

In this section we give an explicit formula of the Fourier coefficients of the Jacobi-Eisenstein series $E_{k+\frac{r+1}{2},D_r} \in J_{k+\frac{r+1}{2},D_r}$ for $k \geq 2$ (see Definition 8.3 and Theorem 8.4) and we prove Theorem 1.5.

8.1. **Cohen type Eisenstein series of level 8.** Let $\mathscr{H}_k$ ($k \in \mathbb{N}$) be the Cohen Eisenstein series of level 4 of weight $k+\frac{1}{2}$ which is defined by

$$\mathscr{H}_k(\tau) \ := \ \zeta(1-2k) + \sum_{N>0} H(k,N)q^N.$$

(see [Co 75, PP. 272–273] for the definition of $H(k,N)$). If $k \geq 2$, then $\mathscr{H}_k \in M_{k+\frac{1}{2}}^+(4)$.

If N is written by $(-1)^k N = Df^2$ with a fundamental discriminant D and $f \in \mathbb{N}$, then $H(k,N)$ satisfies

$$H(k,N) \ = \ H(k,|D|) \sum_{d|f} \mu(d) \left(\frac{D}{d}\right) d^{k-1} \sigma_{2k-1}(f/d) \tag{8.1}$$

and $H(k,|D|) = L(1-k, \left(\frac{D}{*}\right))$, where $L(s,\chi)$ is the Dirichlet L-function.

Let $r \in \{1,3,5,7\}$. We assume $(-1)^k \equiv -r \mod 4$. If k is odd, then $r \in \{1,5\}$, and if k is even, then $r \in \{3,7\}$. We have $k + \frac{r+1}{2} \in 2\mathbb{Z}$.

We define the Cohen type Eisenstein series of level 8 by

$$\mathscr{H}^*_{r,k}(\tau) \quad := \quad \zeta(1-2k) + \sum_{N>0} H^*_r(k,N) q^N,$$

where we set

$$H^*_r(k,N) \quad := \quad H(k,N) + \frac{H(k,N) - H(k,4N)}{2^{k-1}\left(\left(\frac{8}{r}\right) + 2^k\right)}.$$

By definition, we have

$$2^{k-1}\left(\left(\frac{8}{r}\right) + 2^k\right)\mathscr{H}^*_{r,k} = \left(1 + \left(\frac{8}{r}\right)2^{k-1} + 2^{2k-1}\right)\mathscr{H}_k - \mathscr{H}_k|U_k(4),$$

where the operator $U_k(4)$ is explained in §1.4. Since the fact $\mathscr{H}_k|U_k(4) \in M^+_{k+\frac{1}{2}}(8)$ $(k \geq 2)$ follows from [U-Y 10, Prop. 6], we have $\mathscr{H}^*_{r,k} \in M^+_{k+\frac{1}{2}}(8)$ for $k \geq 2$.

The vector space $M^{+,-r}_{k+\frac{1}{2}}(8)$ has been defined in §2.4.

Proposition 8.1. *Assume $k + \frac{r+1}{2}$ is even with $k \geq 2$. Then, we have*

$$M^{+,-r}_{k+\frac{1}{2}}(8) \quad = \quad \mathbb{C}\mathscr{H}^*_{r,k} \oplus S^{+,-r}_{k+\frac{1}{2}}(8).$$

In particular, $\mathscr{H}^*_{r,k} \in M^{+,-r}_{k+\frac{1}{2}}(8)$.

Proof. We assume $N \equiv 4 - r \mod 8$. Since $((-1)^k N)^2 \equiv r^2 - 8 \mod 16$, we obtain $\left(\frac{8}{(-1)^k N}\right) = (-1)^{(((-1)^k N)^2-1)/8} = (-1)^{(r^2-9)/8} = -\left(\frac{8}{r}\right)$. By virtue of the property (8.1), we have

$$H(k,4N) \quad = \quad \left(1 + 2^{2k-1} - \left(\frac{8}{(-1)^k N}\right)2^{k-1}\right) H(k,N). \tag{8.2}$$

Thus we obtain $H^*_r(k,N) = 0$ for $N \equiv 4 - r \mod 8$.

Since $\mathscr{H}^*_{r,k} \in M^+_{k+\frac{1}{2}}(8)$, we obtain $\mathscr{H}^*_{r,k} \in M^{+,-r}_{k+\frac{1}{2}}(8)$.

By virtue of Proposition 6.3 and Lemma 4.1, we have

$$\dim M^{+,-r}_{k+\frac{1}{2}}(8) - \dim S^{+,-r}_{k+\frac{1}{2}}(8) \quad = \quad \dim J_{k+\frac{r+1}{2},D_r} - \dim J^{cusp}_{k+\frac{r+1}{2},D_r} \quad \leq \quad 1$$

and $\mathscr{H}^*_{r,k} \notin S^{+,-r}_{k+\frac{1}{2}}(8)$. Thus this proposition follows. □

Lemma 8.2. *Let N be a positive-integer such that $N \equiv 0, 4, -r \mod 8$. There exists $(n', r') \in \mathbb{Z} \times D^\sharp_r$ such that $N = 8(n' - \beta(r'))$. We write $(-1)^k N = Df^2$ with a fundamental discriminant D and $f \in \mathbb{N}$.*

Then, if $f \equiv 1 \bmod 2$, *we have*

$$H_r^*(k,N) \;=\; \frac{\left(1+\left(\frac{8}{r}\right)\left(\frac{8}{D}\right)\right)H(k,N)}{\left(1+\left(\frac{8}{r}\right)2^k\right)}. \tag{8.3}$$

We remark that if $N \equiv -r \bmod 8$, *then* $f \equiv 1 \bmod 2$ *and* $\left(\frac{8}{D}\right) = \left(\frac{8}{(-1)^k N}\right) = \left(\frac{8}{r}\right)$.

If $f \equiv 0 \bmod 2$, *we have*

$$H_r^*(k,N) \;=\; \frac{H(k,N)+\left(\frac{8}{r}\right)2^k H(k,\frac{N}{4})}{\left(1+\left(\frac{8}{r}\right)2^k\right)}. \tag{8.4}$$

We remark that if $N \equiv 0 \bmod 4$, *then* $D \equiv 0 \bmod 4$ *or* $f \equiv 0 \bmod 2$.

Proof. The formula (8.3) follows from the identity (8.2) in the proof of Proposition 8.1 under the assumption $f \equiv 1 \bmod 2$.

The formula (8.4) follows from the identity

$$\left(1+2^{2k-1}\right)H(k,N) - H(k,4N) \;=\; 2^{2k-1}H\left(k,\frac{N}{4}\right)$$

under the assumption $f \equiv 0 \bmod 2$. □

8.2. **Jacobi-Eisenstein series.**

Definition 8.3. *For* $k \geq 2$ *and for* $r \in \{1,3,5,7\}$ *such that* $k+\frac{r+1}{2}$ *is even, we define*

$$E_{k+\frac{r+1}{2},D_r}(\tau,z) \;:=\; \sum_{A\in SL(2,\mathbb{Z})/\Gamma_\infty}\sum_{\lambda\in D_r} 1|_{k,D_r}(\lambda,0)|_{k,D_r}A,$$

where we put $\Gamma_\infty := \{\pm\left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right) : n \in \mathbb{Z}\}$.

For the convergence of $E_{k+\frac{r+1}{2},D_r}$, see [Aj 15, Prop. 3.3.6] or [Mo 19b, Thm. 2.6].

Remark that the definitions of E_{4,D_5} (case $(r,k) = (5,1)$) and E_{4,D_7} (case $(r,k) = (7,0)$) are not involved in Definition 8.3. The functions E_{4,D_5} and E_{4,D_7} are constructed by using a theta function (see [Mo 19b, §3.3.1], for example. For the Fourier coefficients of E_{4,D_5}, see also Theorem 8.5).

We now obtain an explicit formula of Fourier coefficients of Jacobi Eisenstein series $E_{k+\frac{r+1}{2},D_r}$ of index D_r for $k \geq 2$. As for notation $\beta(r')$, see §4.

Theorem 8.4. *Assume that* $k+\frac{r+1}{2}$ *is an even integer with* $k \geq 2$. *The Jacobi-Eisenstein series* $E_{k+\frac{r+1}{2},D_r} \in J_{k+\frac{r+1}{2},D_r}$ *has the Fourier expansion:*

$$E_{k+\frac{r+1}{2},D_r}(\tau,z) \;=\; \sum_{n'\in\mathbb{Z}}\sum_{r'\in D_r^\sharp} e_{r,k}(n',r')e(n'\tau+\beta(r',z)),$$

where $N = 8(n' - \beta(r'))$ *and*

$$e_{r,k}(n',r') \;=\; \begin{cases} 1, & \text{if } N=0,\\ \dfrac{H_r^*(k,N)}{\zeta(1-2k)}, & \text{if } N>0 \text{ and } N\equiv 0 \mod 4,\\ \dfrac{H_r^*(k,N)}{2\zeta(1-2k)}, & \text{if } N>0 \text{ and } N\equiv -r \mod 8. \end{cases} \tag{8.5}$$

Proof. The function $E_{k+\frac{r+1}{2},D_r}$ (resp. $\mathscr{H}^*_{r,k}$) is an eigenform for Hecke operator $T^J(p)$ (resp. $T(p^2)$) for any odd prime p with the eigenvalue $1+p^{2k-1}$ (see [Aj 15, Thm.3.3.18], for example). Thus, by virtue of Theorems 1.2 and 1.3, both $E_{k+\frac{r+1}{2},D_r}$ and $\mathscr{H}^*_{r,k}$ correspond to E_{2k}, where E_{2k} denotes the usual Eisenstein series in $M_{2k}(1)$ with the constant term 1. Therefore $E_{k+\frac{r+1}{2},D_r}$ and $\mathscr{H}^*_{r,k}$ correspond each other by Proposition 6.3. We obtain $\mathscr{J}^{even}_{r,k}(E_{k+\frac{r+1}{2},D_r}) = c\mathscr{H}^*_{r,k}$ with a certain $c \in \mathbb{C}$. By comparing the constant terms, we have $c = \zeta(1-2k)^{-1}$. The formula of $e_{r,k}(n',r')$ in (8.5) follows from the definition of the map $\mathscr{J}^{even}_{r,k}$ (see (6.3)).

□

Remark $\dim J_{4,D_5} = 1$. There exists a $E_{4,D_5} \in J_{4,D_5}$ which corresponds $E_2^{(2)}$ by Theorem 1.4. Such a function E_{4,D_5} in J_{4,D_5} has been defined in [Mo 19b]. The explicit formula of Fourier coefficients of E_{4,D_5} is written in [Mo 19b, PP. 76–77]. We quote this result. By using the identity

$$\#\left\{(x_1,x_2,x_3)\in\mathbb{Z}^3 \;:\; -2D = \sum_{i=1}^{3}(x_i^2+x_i)+\frac{3}{4}\right\} \;=\; r_3(-8D)$$

for $-8D \in \mathbb{N}$ such that $-8D \equiv 3 \mod 8$, we have the following theorem.

Theorem 8.5 ([B-S 23, Mo 19b]). *Let* $E_{4,D_5}(\tau,z) = \sum_{n',r'} e_{5,1}(n',r')e(n'\tau+\beta(r',z))$ *be the Fourier expansion of* $E_{4,D_5} \in J_{4,D_5}$. *Then,*

$$e_{5,1}(n',r') \;=\; \begin{cases} r_3(N), & \text{if } N\equiv 0 \mod 4,\\ \frac{1}{2}r_3(N), & \text{if } N\equiv 3 \mod 8, \end{cases}$$

where $N = 8(n'-\beta(r'))$.

We remark that there is a misprint in the formula of $C_{4,n}(D,r)$ in [Mo 19b, P. 77]. Because of the condition $\sum_{i=1}^{8} x_i \in 2\mathbb{Z}$ of the lattice E_8, the term

$$\#\left\{x\in\mathbb{Z}^{8-n} \;:\; -2D = x_1^2+x_1+\cdots+x_{8-n}^2+x_{8-n}+\frac{8-n}{4}\right\}$$

in [Mo 19b, P. 77] must be halved.

We will show $r_3(N) = 12(H(1,4N)-2H(1,N))$ in §8.3. By using this identity, since $\zeta(-1) = -\frac{1}{12}$ and since $H_5^*(1,N) = 2H(N)-H(4N) = -\frac{1}{12}r_3(N)$, Theorem 8.4 is also valid for $(r,k)=(5,1)$ by Theorem 8.5.

8.3. **Zagier type Eisenstein series of level** 8**.** We now consider the case $k = 1$ of $\mathscr{H}_k^*$. We define

$$\mathscr{H}_1(\tau) \quad := \quad -\frac{1}{12} + \sum_{N>0} H(N) q^N,$$

where we set $H(N) := H(1, N)$. We define the Zagier Eisenstein series $\mathscr{F}$ of weight $3/2$ of level 4 by

$$\mathscr{F}(\tau) \quad := \quad \mathscr{H}_1(\tau) + \frac{1}{16\pi\sqrt{v}} \sum_{n\in\mathbb{Z}} \alpha(n^2 y) q^{-n^2},$$

where $v = \mathrm{Im}(\tau)$ and where we put $\alpha(t) := \int_1^\infty e^{-4\pi u t} u^{-3/2}\, du$. By a straight-forward calculation, we have

$$\mathscr{F}|U_1(4) - 2\mathscr{F} \quad = \quad \mathscr{H}_1|U_1(4) - 2\mathscr{H}_1 \quad = \quad \frac{1}{12} + \sum_{\substack{N>0 \\ N\equiv 0,3 \mod 4}} (H(4N) - 2H(N)) q^N.$$

We define

$$\mathscr{H}_1^*(\tau) \quad := \quad \mathscr{H}_{5,1}^*(\tau) \quad = \quad -(\mathscr{F}|U_1(4) - 2\mathscr{F}) \quad = \quad -\frac{1}{12} + \sum_{\substack{N>0 \\ N\equiv 0,3 \mod 4}} (2H(N) - H(4N)) q^N.$$

We now prove Theorem 1.5.

Proof of Theorem 1.5. First, we show the identity

$$r_3(N) \quad = \quad 12(H(4N) - 2H(N)) \tag{8.6}$$

for any $N \in \mathbb{N} \cup \{0\}$. Here we defined $H(N) = 0$ for $N \in \mathbb{N}$ such that $-N \equiv 2, 3 \bmod 4$.

The identity $r_3(4N) = r_3(N)$ follows from the definition of r_3. For N such that $-N \equiv 0, 1 \bmod 4$, the identity $H(16N) - 2H(4N) = H(4N) - 2H(N)$ follows from the property (8.1). Furthermore, we obtain

$$H(4N) - 2H(N) \quad = \quad \begin{cases} H(N), & \text{if } N \equiv 0 \mod 4 \text{ and } \frac{N}{4} \equiv 1, 2 \mod 4, \\ 2H(N), & \text{if } N \equiv 3 \mod 8, \\ 0, & \text{if } N \equiv 7 \mod 8. \end{cases}$$

It is shown in [Co 75, P. 274] that the right hand side of the above identity equals to $\frac{1}{12} r_3(N)$ for N which satisfies the above conditions. Therefore, for $-N \equiv 0, 1 \bmod 4$, the identity (8.6) holds.

If $-N \equiv 2, 3 \bmod 4$, then $H(4N) - 2H(N) = H(4N) = \frac{1}{12} r_3(4N) = \frac{1}{12} r_3(N)$. Thus, for any $N \in \mathbb{N}$, the identity (8.6) holds. If $N = 0$, then $H(4N) - 2H(N) = -H(0) = \frac{1}{12} = \frac{1}{12} r_3(0)$. We conclude the identity (8.6) for any $N \in \mathbb{N} \cup \{0\}$.

We now have

$$(\theta^3|U_1(4))(\tau) = \sum_{\substack{N\geq 0 \\ N\equiv 0,3 \mod 4}} r_3(N)q^N = 12 \sum_{\substack{N\geq 0 \\ N\equiv 0,3 \mod 4}} (H(4N)-2H(N))q^N$$
$$= -12\mathscr{H}_1^*(\tau).$$

Remark that the constant term of $\theta^3|U_1(4)$ is $r_3(0)=1$.

By virtue of Proposition 6.3 and Theorem 8.5, we have

$$\theta^3|U_1(4) = \sum_{N\equiv 0,3,4 \mod 8} r_3(N)q^N = \mathcal{J}_{5,1}^{even}(E_{4,D_5}) \in M_{3/2}^{+,-5}(8).$$

Since $\dim M_{3/2}^{+,-5}(8) = \dim J_{4,D_5} = 1$ and since the constant terms of $\theta^3|U_1(4)$ and of $E_{3/2}^{(8)}$ are both 1, we have $-12\mathscr{H}_1^*(\tau) = \theta^3|U_1(4) = E_{3/2}^{(8)} \in M_{3/2}^{+,-5}(8)$. Therefore, we conclude the proof of Theorem 1.5.

We remark that, since $\theta^3 \in M_{3/2}(4)$, the fact $\theta^3|U_1(4) \in M_{3/2}^{+}(8)$ follows from [U-Y 10, Prop. 6]. Since $r_3(N)=0$ for $N\equiv 7 \mod 8$, the fact $\theta^3|U_1(4)\in M_{3/2}^{+,-5}(8)$ also follows. □

9. Maps from Jacobi forms to elliptic modular forms

In this section we prove Theorems 1.8 and 1.9.

Theorems 1.8 and 1.9 follow essentially from the facts that the Hecke eigenvalues of $\phi \in J_{k+\frac{r+1}{2},D_r}$ coincide with the Hecke eigenvalues of $f \in M_{2k}^{new}(N)$ $(N=1,2)$, if ϕ corresponds to f by Theorem 1.2, 1.3 or 1.4. And the Hecke eigenvalues of f coincide with the Fourier coefficients of f up to a constant.

Proof of Theorem 1.8. We assume $\phi \in J_{k+\frac{r+1}{2},D_r}$ and assume that ϕ is an Hecke eigenform. We put $g = \mathcal{J}_{r,k}^{even}(\phi) \in M_{k+\frac{1}{2}}^{+,-r}(8)$ and take the Fourier expansion $g = \sum A(m)q^m$.

For any odd prime p and for any discriminant m, due to (6.12), we have

$$\lambda_p A(m) = A(p^2m) + \left(\frac{(-1)^k m}{p}\right) p^{k-1}A(m) + p^{2k-1}A(m/p^2), \tag{9.1}$$

where λ_p denotes the Hecke eigenvalue of ϕ with respect to $T^J(p)$. Thus, if we assume $p^2 \not| m$, then

$$\begin{aligned}
&\sum_{u=0}^{\infty}\sum_{l=0}^{u} p^{l(k-1)}\left(\frac{(-1)^k m}{p^l}\right) A(p^{2(u-l)}m)X^u \\
=\ & (1-p^{k-1}\left(\frac{(-1)^k m}{p}\right)X)^{-1}\sum_{t=0}^{\infty} A(p^{2t}m)X^t \\
=\ & (1-p^{k-1}\left(\frac{(-1)^k m}{p}\right)X)^{-1}\frac{1-p^{k-1}\left(\frac{(-1)^k m}{p}\right)X}{1-\lambda_p X+p^{2k-1}X^2}A(m) \\
=\ & \frac{A(m)}{1-\lambda_p X+p^{2k-1}X^2}.
\end{aligned}$$

Since λ_p is the Hecke eigenvalue of ϕ, the value λ_p is the Hecke eigenvalue of $f \in M_{2k}^{new,\epsilon_2}(2) \oplus M_{2k}^{\epsilon_1}(1)$, where f is a modular form which corresponds to ϕ by the isomorphism in Theorem 1.1. Without loss of generality, we can assume $f = \sum a_f(n)q^n$ with $a_f(1) = 1$. Then,

$$\frac{A(m)}{1-\lambda_p X+p^{2k-1}X^2} = A(m)\sum_{l=0}^{\infty} a_f(p^l)X^l.$$

Thus

$$\sum_{l=0}^{u} p^{l(k-1)}\left(\frac{(-1)^k m}{p^l}\right) A(p^{2(u-l)}m) = A(m)a_f(p^l). \tag{9.2}$$

Since $a_f(*)$ is a multiplicative function, by assuming m is not divisible by p^2 for any odd prime p, we obtain

$$\sum_{d|n} d^{k-1}\left(\frac{(-1)^k m}{d}\right) A(n^2d^{-2}m) = A(m)a_f(n) \tag{9.3}$$

for any odd integer n.

We now need to show (9.3) for any even integer n. Since S_{d_0} is a linear map, it is sufficient to consider three cases that ϕ is in $J^{cusp,new}_{k+\frac{r+1}{2},D_r}$, ϕ is in $J^{cusp,old}_{k+\frac{r+1}{2},D_r}$ or ϕ is a Jacobi-Eisenstein series $E_{k+\frac{r+1}{2},D_r}$.

First, we consider the case $\phi \in J^{cusp,new}_{k+\frac{r+1}{2},D_r}$ and we show $S_{d_0}(\phi) \in S_{2k}^{new,\epsilon_2}(2)$. In this case f belongs to $S_{2k}^{new,\epsilon_2}(2)$. Then $g|U_k(4) = \lambda_2 g$, where $g = \mathcal{J}^{even}_{r,k}(\phi)$, and where $\lambda_2 = a_f(2)$ by Theorem 6.7. Due to the definition of $U_k(4)$, we have $A(4^l m) = \lambda_2 A(4^{l-1}m) = \lambda_2^l A(m) = a_f(2)^l A(m)$ for any integer m.

By assumption, $(-1)^k d_0$ is a fundamental discriminant with $d_0 \equiv 0 \mod 4$. Thus $\left(\frac{(-1)^k d_0}{2}\right) = \left(\frac{8}{(-1)^k d_0}\right) = 0$ by the definition of the Kronecker symbol. Therefore, for any odd integer n and for any natural number l, we have

$$\begin{aligned} a_f(2^l n)A(d_0) &= a_f(2)^l \sum_{d|n} d^{k-1} \left(\frac{(-1)^k d_0}{d}\right) A(n^2 d^{-2} d_0) \\ &= \sum_{d|2^l n} d^{k-1} \left(\frac{(-1)^k d_0}{d}\right) A((2^l n)^2 d^{-2} d_0). \end{aligned}$$

Thus $S_{d_0}(\phi) = A(d_0) f \in S_{2k}^{new,\epsilon_2}(2)$. The case $\phi \in J_{k+\frac{r+1}{2},D_r}^{cusp,new}$ is completed.

Next, we consider the case $\phi \in J_{k+\frac{r+1}{2},D_r}^{cusp,old}$ and we show $S_{d_0}(\phi) \in S_{2k}^{\epsilon_1}(1)$.

In this case f belongs to $S_{2k}^{\epsilon_1}(1)$. A Fourier-Jacobi coefficient of the Ikeda lift of f is ϕ (see §3). We take a Hecke eigenform $h = \sum_{n\equiv 0,(-1)^k \bmod 4} b(n) q^n \in S_{k+\frac{1}{2}}^{+}(4)$ which corresponds to f by the Shimura correspondence. Let λ_2 be the Hecke eigenvalue of h with respect to the Hecke operator $T(2^2)$. We put $h^* = \sum_{n\equiv 0,(-1)^k \bmod 4} b^*(n) q^n$, where we denote

$$b^*(n) := b(4n) - b(n)\left\{\left(\frac{8}{r}\right) 2^{k-1} + \lambda_2\right\}. \tag{9.4}$$

Since h^* is a linear combination of h and $h|U_k(4)$, due to [U-Y 10], we have $h^* \in S_{k+\frac{1}{2}}^{+}(8)$. Since h is a Hecke eigenform, we have

$$\lambda_2 b(n) = b(4n) + \left(\frac{(-1)^k n}{2}\right) 2^{k-1} b(n) + 2^{2k-1} b(n/4). \tag{9.5}$$

If $n \equiv 4 - r \mod 8$, then $\left(\frac{(-1)^k n}{2}\right) = \left(\frac{(-1)^{\frac{r+1}{2}}(4-r)}{2}\right) = -\left(\frac{8}{r}\right)$ and $b(n/4) = 0$. Thus $b^*(n) = 0$ for n such that $n \equiv 4 - r \mod 8$. Therefore, h^* belongs to $S_{k+\frac{1}{2}}^{+,-r}(8)$.

By using the identities (9.4) and (9.5), it is straight-forward to show the identities

$$2b^*(4n) - 2^{k-1}\left(\frac{8}{r}\right) b^*(n) = \lambda_2 b^*(n)$$

for n such that $n \equiv -r \mod 8$, and

$$b^*(4n) + 2^{2k-1} b^*(n/4) = \lambda_2 b^*(n) \tag{9.6}$$

for n such that $n \equiv 0 \mod 4$.

The Hecke eigenvalue λ_2 of ϕ at $p = 2$ is also the Hecke eigenvalue of f at $p = 2$. Thus $\lambda_2 = a_f(2)$.

We recall the assumption that $(-1)^k d_0$ is a fundamental discriminant with $d_0 \equiv 0 \mod 4$. Thus $\left(\frac{(-1)^k d_0}{2}\right) = 0$. Since the identity (9.6) is similar to (9.1), by using (9.6) and by a similar argument to (9.2), we obtain

$$(9.7)\qquad \sum_{l=0}^{u} 2^{l(k-1)} \left(\frac{(-1)^k d_0}{2^l}\right) b^*(2^{2(u-l)} d_0) \;=\; b^*(d_0) a_f(2^l),$$

where $\left(\frac{(-1)^k d_0}{2^l}\right) = 1$ or 0 according as $l = 0$ or not.

The functions h^*, h and ϕ have the same Hecke eigenvalue for any odd prime p. There exists a $\psi \in J^{cusp}_{k+\frac{r+1}{2},D_r}$ such that $\mathcal{J}^{even}_{r,k}(\psi) = h^*$. Then ϕ and ψ have the same Hecke eigenvalue for any odd prime p. By the strong multiplicity one theorem for $S_{2k}(2)$ and by Theorem 1.1, ψ coincides with ϕ up to a constant multiple. Hence $g(= \mathcal{J}^{even}_{r,k}(\phi))$ coincides with $h^*(= \mathcal{J}^{even}_{r,k}(\psi))$ up to a constant multiple. Thus there exists a constant c such that $A(m) = c\, b^*(m)$ for any $m \in \mathbb{N}$. Hence the identity (9.7) also holds if we replace $b^*(*)$ by $A(*)$. Thus, by a similar argument to the case of $\phi \in J^{cusp,new}_{k+\frac{r+1}{2},D_r}$, the identity (9.3) holds for any $n \in \mathbb{N}$ and for any fundamental discriminant $m = (-1)^k d_0$ such that $d_0 \equiv 0 \bmod 4$. Therefore we conclude $S_{d_0}(\phi) = A(d_0) f \in S^{\epsilon_1}_{2k}(1)$ for $\phi \in J^{cusp,old}_{k+\frac{r+1}{2},D_r}$.

Finally, we consider the case $\phi = E_{k+\frac{r+1}{2},D_r}$. We write

$$E_{k+\frac{r+1}{2},D_r}(\tau, z) \;=\; \sum_{n' \in \mathbb{Z}} \sum_{r' \in D_r^\sharp} e_{r,k}(n', r') e(n'\tau + \beta(r, z)).$$

The explicit formula of $e_{r,k}(n', r')$ has been obtained in Theorem 8.4.

If $(r, k) \neq (5, 1)$, then $E_{k+\frac{r+1}{2},D_r}$ corresponds to the Eisenstein series $E_{2k} \in M_{2k}(1)$. This fact follows from Proposition 3.4. If $(r, k) = (5, 1)$, then E_{4,D_5} corresponds to $E^{(2)}_2 \in M^{new}_2(2)$. This fact follows from Theorem 1.4. For any pair (r, k), by the same argument to the above, the identity (9.3) follows for $g = \mathcal{J}^{even}_{r,k}(E_{k+\frac{r+1}{2},D_r}) = \sum A(m) q^m$, for any $n \in \mathbb{N}$ and for any fundamental discriminant $(-1)^k d_0$ with $d_0 \equiv 0 \bmod 4$.

We consider the case $(r, k) \neq (5, 1)$. The case $(r, k) = (5, 1)$ follows similar.

We put $\mathbb{G}_{2k} = \frac{\zeta(1-2k)}{2} E_{2k}$. We take the Fourier expansion of $\mathbb{G}_{2k}$:

$$\mathbb{G}_{2k}(\tau) \;=\; \frac{\zeta(1-2k)}{2} + \sum_{n \geq 1} \sigma_{2k-1}(n) q^n,$$

where $\sigma_m(n) = \sum_{d|n} d^m$. Then, the Fourier coefficient of $S_{d_0}(E_{k+\frac{r+1}{2},D_r})$ at q^n is $A(d_0)\sigma_{2k-1}(n)$ for any $n \geq 1$ due to the identity (9.3).

To show the identity $S_{d_0}(E_{k+\frac{r+1}{2},D_r}) = A(d_0)\mathbb{G}_{2k}$, it is sufficient to show that the constant term of both sides coincides.

The constant term of $S_{d_0}(E_{k+\frac{r+1}{2},D_r})$ is $\frac{A(0)}{2(1+\left(\frac{8}{r}\right)2^k)} L(1-k, \left(\frac{(-1)^k d_0}{*}\right))$ due to the definition of the map S_{d_0}. On the other hand, the constant term of $A(d_0)\mathbb{G}_{2k}$ is $A(d_0)\frac{\zeta(1-2k)}{2}$.

Thus we need to show

$$(9.8)\qquad \frac{A(d_0)}{A(0)} = \frac{L(1-k,\left(\frac{(-1)^k d_0}{*}\right))}{(1+\left(\frac{8}{r}\right)2^k)\zeta(1-2k)}.$$

We recall that $A(0) = e_{r,k}(0,0)$ and $A(d_0) = e_{r,k}(n',r')$ for $d_0 = 8(n'-\beta(r'))$. By virtue of Theorem 8.4 and Lemma 8.2, and by assumption on d_0, we have $A(0) = 1$ and $A(d_0) = \frac{H(k,(-1)^k d_0)}{(1+\left(\frac{8}{r}\right)2^k)\zeta(1-2k)}$. Since, $H(k,(-1)^k d_0) = L(1-k,\left(\frac{(-1)^k d_0}{*}\right))$, the identity (9.8) holds. Therefore we have $S_{d_0}(E_{k+\frac{r+1}{2},D_r}) = A(d_0)\mathbb{G}_{2k} \in M_{2k}(1)$.

Since S_{d_0} is a linear map, for any $\phi \in J_{k+\frac{r+1}{2},D_r} = J^{cusp,new}_{k+\frac{r+1}{2},D_r} \oplus J^{cusp,old}_{k+\frac{r+1}{2},D_r} \oplus \mathbb{C}E_{k+\frac{r+1}{2},D_r}$ we obtain $S_{d_0}(\phi) = A(d_0)f$ with a $f \in M^{new,\epsilon_2}_{2k}(2) \oplus M^{\epsilon_1}_{2k}(1)$. We conclude Theorem 1.8. □

Proof of Theorem 1.9. The proof is similar to the proof of Theorem 1.8. By virtue of the identity (5.5), we have

$$\lambda_p A(m) = \left(\frac{-4}{p}\right)\left\{A(p^2m) + \left(\frac{(-1)^k m}{p}\right)p^{k-1}A(m) + p^{2k-1}A(m/p^2)\right\},$$

for $m \equiv -r \mod 8$ and for any odd prime p, where λ_p denotes the Hecke eigenvalue of ϕ for $T^J(p)$.

Let $f \in S^{new,\epsilon_2}_{2k}(2)$ be a Hecke eigenform which corresponds to $\phi \in J^{cusp,new}_{k+\frac{r+1}{2},D_r}$ by Theorem 1.3. We take the Fourier expansion $f = \sum a_f(n)q^n$ and assume $a_f(1) = 1$.

By an argument analogous to the calculation of (9.3), if m is not divisible by p^2 for any odd prime p, we have

$$\left(\frac{-4}{n_1}\right)\sum_{d|n_1} d^{k-1}\left(\frac{(-1)^k m}{d}\right)A(n^2d^{-2}m) = A(m)a_f(n_1)$$

for any odd integer n_1.

On the other hand, since f belongs to $S^{new,\epsilon_2}_{2k}(2)$, we have $f|_{2k}\left(\begin{smallmatrix}0 & -1\\ 2 & 0\end{smallmatrix}\right) = \epsilon_2 i^{-2k}f$. And it is known that $f|_{2k}\left(\begin{smallmatrix}0 & -1\\ 2 & 0\end{smallmatrix}\right) = -2^{1-k}a_f(2)f$ (see [Zh 13, Introduction]). Thus we have

$$a_f(2) = -\epsilon_2 i^{-2k}2^{k-1} = \left(\frac{-8}{r}\right)(-1)^k 2^{k-1} = \left(\frac{-8}{r}\right)(-1)^{\frac{r-1}{2}}2^{k-1} = \left(\frac{8}{r}\right)2^{k-1}.$$

Here we used the assumption $k + \frac{r+1}{2} \equiv 1 \mod 2$. The arithmetic function $a_f(*)$ is multiplicative and the identity $a_f(2^{e_2}) = a_f(2)^{e_2}$ holds. Therefore,

$$\begin{aligned}\left(\left(\frac{8}{r}\right)2^{k-1}\right)^{e_2}\left(\frac{-4}{n_1}\right)\sum_{d|n_1} d^{k-1}\left(\frac{(-1)^k d_0}{d}\right)A(n^2d^{-2}d_0) &= A(d_0)a(2^{e_2})a(n_1)\\ &= A(d_0)a(2^{e_2}n_1)\end{aligned}$$

for any fundamental discriminant $(-1)^{k-1}d_0$ such that $d_0 \equiv -r \mod 8$ and $d_0 > 0$.

We conclude $S_{d_0}(\phi) = A(d_0)f \in S_{2k}^{new,\epsilon_2}(2)$ for any Hecke eigenform $\phi \in J_{k+\frac{r+1}{2},D_r}^{cusp,new}$. Since the map S_{d_0} is linear, we conclude Theorem 1.9. $\square$

References

[Aj 15] A. Ajouz: Hecke operators on Jacobi forms of lattice index and the relation to elliptic modular forms, Ph.D. thesis, University of Siegen.
URL: https://www.researchgate.net/publication/285601134

[Ar 98] T. Arakawa: Köcher-Maass Dirichlet series corresponding to Jacobi forms and Cohen Eisenstein series *Comment. Math. Univ. St. Paul.* **47** (1998), no. 1, 93–122.

[Boe 83] S. Boecherer: Über die Fourier-Jacobi-Entwicklung Siegelscher Eisensteinreihen, *Math. Z.* **183** (1983), 21–46.

[Boy 11] H. Boylan: Jacobi forms, quadratic modules and Weil representations over number Fields. PhD thesis, (2011). URL: http://dokumentix.ub.uni-siegen.de/opus/volltexte/2012/597/.

[B-S 23] H. Boylan and N.-P. Skoruppa: Jacobi forms of lattice index I. Basic theory. Preprint (2023) arXiv:2309.04738.

[B-K 00] S. Breulmann and M. Kuss: On a conjecture of Duke-Imamoḡlu. *Proc. Amer. Math. Soc.* **128** (2000), no. 6, 1595–1604.

[Br 06] K. Bringmann: Lifting maps from a vector space of Jacobi cusp forms to a subspace of elliptic modular forms, *Math. Z.* **253** (2006), no. 4, 735–752.

[Co 75] H. Cohen: Sums involving the values at negative integers of L-functions of quadratic characters, *Math. Ann.* **217**, (1975), 271–285.

[Ha 11] S. Hayashida : Fourier-Jacobi expansion and the Ikeda lift, *Abh. Math. Semin. Univ. Hambg.* **81** (2011), 1–17.

[Ik 01] T. Ikeda : On the lifting of elliptic cusp forms to Siegel cusp forms of degree 2n, *Ann. of Math. (2)* **154**, no.3 (2001), 641–681.

[Ko 80] W. Kohnen : Modular forms of half integral weight on $\Gamma_0(4)$, *Math, Ann.* **248** (1980), 249–266.

[Mi 89] T. Miyake: Modular Forms, Springer-Verlag, Berlin, (1989).

[Mo 19a] A. Mocanu: Poincaré and Eisenstein series for Jacobi forms of lattice index. *J. Number Theory* **204** (2019), 296–333.

[Mo 19b] A. Mocanu: On Jacobi forms of lattice index, Ph.D. thesis, University of Nottingham (2019), 158 pages.

[Mu 89] A. Murase: L-functions attached to Jacobi forms of degree n. I. The basic identity., J. Reine Angew. Math. **401** (1989), 122–156.

[S-Z 88] N.-P. Skoruppa and D. Zagier: Jacobi forms and a certain space of modular forms, Invent. Math. **94**, (1988), 113–146.

[U-Y 10] M. Ueda and S. Yamana: On newforms for Kohnen plus spaces. *Math. Z.* **264** (2010), no. 1, 1–13.

[Ya 14] Y. Yang: Modular forms of half-integral weights on SL(2,$\mathbb{Z}$), Nagoya Math. J. **215** (2014), 1–66.

[Za 75] D. Zagier: Nombres de classes et formes modulaires de poids 3/2, *C. R. Acad. Sci. Paris Sér. A-B 281* (1975), A883–A886.

[Zh 13] Y. Zhang: Eigenvalue of Fricke involution on new forms of level 4 and of trivial character, Preprint (2013) arXiv:1308.3417v1.

[Zi 89] C. Ziegler: Jacobi forms of higher degree, *Abh. Math. Sem. Univ. Hamburg.* **59** (1989), 191–224.

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