A conjecture on the zeta functions of pairs of ternary quadratic forms

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June 2017

Pairs of ternary quadratic forms

Let V be the vector space of pairs of ternary quadratic forms over $\mathbb{C}$ in three variables $v = (v_1, v_2, v_3)$. For $x = (x_1, x_2) \in V$, write

$$x_k(v) = \sum_{1 \le i \le j \le 3} x_{k,ij} v_i v_j \qquad (k = 1, 2)$$

We identify x_k by symmetric matrix as usual. We define a binary cubic form $F_x(u)$ in variables $u = (u_1, u_2)$ by

$$F_x(u) = 4 \det(u_1 x_1 - u_2 x_2) = a u_1^3 + b u_1^2 u_2 + c u_1 u_2^2 + d u_2^3.$$

The coefficients a, b, c, d are homogeneous polynomials of $x_{k,ij}$ of degree 3. Put

$$\mathrm{Disc}(x) = \mathrm{Disc}(F_x) = 18abcd + b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2.$$

$\mathrm{Disc}(x)$ is a homogeneous polynomial of $x_{k,ij}$ of degree 12.

Group action

The group $G = \mathrm{SL}_3(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$ acts on V:
$g = (g_1, g_2) \in G$, $x = (x_1, x_2) \in V$,

$$g \cdot x = (p(g_1 x_1) + q(g_1 x_2), r(g_1 x_1) + s(g_1 x_2)),$$

$g_2 = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, $(g_1 x_k)(v) = x_k(v g_1)$. Then we have

$$\mathrm{Disc}(g \cdot x) = (\det g_1)^8 (\det g_2)^6 \, \mathrm{Disc}(x).$$

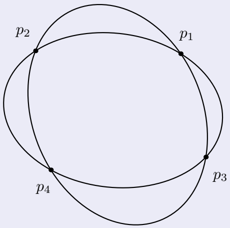
Put $S = \{x \in V \ : \ \mathrm{Disc}(x) = 0\}$. Then S is an irreducible hypersurface and $V \setminus S$ is a single G-orbit. Hence (G, V) is a prehomogeneous vector space and $\mathrm{Disc}(x)$ is its fundamental relative invariant.

$\mathrm{Zero}(x)$

For any $x \in V \setminus S$, put

$$\mathrm{Zero}(x) = \{v \in \mathbb{P}^2(\mathbb{C}) \,:\, x_1(v) = x_2(v) = 0\}.$$

Then $\mathrm{Zero}(x)$ is a finite set consisting of four points.



The set $V_{\mathbb{R}} \setminus S_{\mathbb{R}}$ decomposes into three $G_{\mathbb{R}}$-orbits V_1, V_2, V_3. V_1, V_2 and V_3 are the set of $x \in V_{\mathbb{R}} \setminus S_{\mathbb{R}}$ such that the cardinality of $\mathrm{Zero}(x) \cap \mathbb{P}^2(\mathbb{R})$ equals 4, 2, and 0, respectively.

Dual lattice

Let $L \subset V_{\mathbb{R}}$ be the lattice of pairs of integral ternary quadratic forms. For $x = (x_1, x_2), y = (y_1, y_2) \in V$, put $\langle x, y \rangle = \mathrm{Tr}(x_1 y_2) + \mathrm{Tr}(x_2 y_1)$ (Tr is the trace of matrices). Then the dual lattice $\hat{L}$ of L with respect to this pairing is the set of pairs of integral symmetric matrices of degree 3. Put $\Gamma_1 = \mathrm{SL}_3(\mathbb{Z})$, $\Gamma_2 = \mathrm{GL}_2(\mathbb{Z})$. The group $\Gamma = \Gamma_1 \times \Gamma_2$ acts on L and $\hat{L}$. For any $y = (y_1, y_2) \in \hat{L}$, put

$$\hat{F}_y(u) = \det(u_1 y_1 - u_2 y_2) = (1/4) F_y(u).$$

$\hat{F}_y(u)$ is an integral binary cubic form. We put

$$\mathrm{Disc}^*(y) = \mathrm{Disc}(\hat{F}_y) = 2^{-8}\,\mathrm{Disc}(F_y).$$

Zeta functions

We denote by Γ_x the isotropy group of $x \in L \setminus S$ in Γ. Then Γ_x is a finite group of order at most 72. Put $\mu(x) = 1/|\Gamma_x|$.
The zeta functions $\xi_i(L, s)$, $\xi_i(\hat{L}, s)$ $(i = 1, 2, 3)$ are define by

$$\xi_i(L, s) = \sum_{x \in \Gamma \backslash L \cap V_i} \frac{\mu(x)}{|\operatorname{Disc}(x)|^s} = \sum_{n=1}^{\infty} \frac{a_i((-1)^{i-1} n)}{n^s},$$

$$\xi_i(\hat{L}, s) = \sum_{y \in \Gamma \backslash \hat{L} \cap V_i} \frac{\mu(y)}{|\operatorname{Disc}^*(y)|^s} = \sum_{n=1}^{\infty} \frac{\hat{a}_i((-1)^{i-1} n)}{n^s},$$

where

$$a_i(n) = \sum_{\substack{x \in \Gamma \backslash (L \cap V_i) \\ \operatorname{Disc}(x) = n}} \mu(x), \qquad \hat{a}_i(n) = \sum_{\substack{y \in \Gamma \backslash (\hat{L} \cap V_i) \\ \operatorname{Disc}^*(y) = n}} \mu(y).$$

Functional equations

The zeta functions $\xi_i(L, s)$, $\xi_i(\hat{L}, s)$ converge absolutely for $\Re(s) > 1$. This was proved in Yukie's book ('*Shintani Zeta functions*', Cambridge Univ. Press, 1993). The functional equations

$$
\begin{aligned}
&(\xi_i(L, 1-s)) \\
&= \Gamma(s)^4 \Gamma\left(s - \frac{1}{6}\right)^2 \Gamma\left(s + \frac{1}{6}\right)^2 \Gamma\left(s - \frac{1}{4}\right)^2 \Gamma\left(s + \frac{1}{4}\right)^2 \\
&\times 2^{8s} 3^{6s} \pi^{-12s} (u^*_{ji}(s))(\xi_j(\hat{L}, s)).
\end{aligned}
$$

were proved by Sato-Shintani (On zeta functions associated with prehomogeneous vector spaces, *Ann. of Math.* **100** (1974), 131–170). Here $u^*_{ji}(s)$' are polynomials of $q = \exp(\pi\sqrt{-1}s)$ and q^{-1} of degree at most 6.

Conjecture

We present the following conjecture which is a quartic analogue of Ohno conjecture (Y. Ohno, A conjecture on coincidence among the zeta functions associated with the space of binary cubic forms, *Amer. J. Math*. **119** (1997), 1083–1094.
J. Nakagawa, On the relations among the class numbers of binary cubic forms, Invent. math. 134, 101-138 (1998)).

Conjecture 1.1

$$\begin{aligned}
\xi_1(\hat{L}, s) &= \xi_1(L, s) + \xi_3(L, s), \\
\xi_2(\hat{L}, s) &= 2\xi_2(L, s), \\
\xi_3(\hat{L}, s) &= 3\xi_1(L, s) - \xi_3(L, s).
\end{aligned}$$

Results

By a ring of rank n we mean a commutative ring with unit that is free of rank n as a $\mathbb{Z}$-module. It is called *nondegenerate* if its discriminant is non-zero.

For any integral binary cubic form $F(u) = au_1^3 + bu_1^2u_2 + cu_1u_2^2 + du_2^3$, we denote by $R(F)$ the cubic ring associated with $F(u)$ by Delone-Faddeev correspondence. $R(F)$ is a free $\mathbb{Z}$-module having $\mathbb{Z}$-basis $\{1, \omega, \theta\}$ and the multiplication table

$$\begin{aligned} \omega^2 &= -ac + b\omega - a\theta, \\ \theta^2 &= -bd + d\omega - c\theta, \\ \omega\theta &= -ad. \end{aligned} \tag{1.1}$$

The correspondence $F \mapsto R(F)$ gives a discriminant preserving bijection between the set of $\mathrm{GL}_2(\mathbb{Z})$-orbits of integral binary cubic forms and the set of isomorphism classes of cubic rings.

For any nondegenerate cubic ring $\mathcal{O}$, we denote by $L(\mathcal{O})$ and $\hat{L}(\mathcal{O})$ the set of $x \in L$ such that $R(F_x) \cong \mathcal{O}$ and the set of $y \in \hat{L}$ such that $R(\hat{F}_y) \cong \mathcal{O}$, respectively. Further we put $L_i(\mathcal{O}) = L(\mathcal{O}) \cap V_i$, $\hat{L}_i(\mathcal{O}) = \hat{L}(\mathcal{O}) \cap V_i$ $(i = 1, 2, 3)$. For any étale algebra over $\mathbb{Q}$, we denote by $\mathcal{O}_k$ the maximal order of k.

Theorem 1

Let k be a cubic field and $\mathcal{O}$ be an order of k such that the index $(\mathcal{O}_k : \mathcal{O})$ is odd and square free. Then the following relations hold:

$$\sum_{y \in \Gamma \backslash \hat{L}_1(\mathcal{O})} \mu(y) = \sum_{x \in \Gamma \backslash L_1(\mathcal{O})} \mu(x) + \sum_{x \in \Gamma \backslash L_3(\mathcal{O})} \mu(x) \quad (\mathrm{Disc}(k) > 0),$$

$$\sum_{y \in \Gamma \backslash \hat{L}_2(\mathcal{O})} \mu(y) = 2 \sum_{x \in \Gamma \backslash L_2(\mathcal{O})} \mu(x) \quad (\mathrm{Disc}(k) < 0),$$

$$\sum_{y \in \Gamma \backslash \hat{L}_3(\mathcal{O})} \mu(y) = 3 \sum_{x \in \Gamma \backslash L_1(\mathcal{O})} \mu(x) - \sum_{x \in \Gamma \backslash L_3(\mathcal{O})} \mu(x) \quad (\mathrm{Disc}(k) > 0).$$

By Theorem 1 and applying Gauss's genus theory on the 2-rank of the ideal class groups of quadratic fields for the reducible algebras $k = \mathbb{Q} \oplus \mathbb{Q}(\sqrt{n})$, we obtain

Theorem 2

If n is a discriminant of a quadratic field, then the following relations hold:

$$\begin{aligned} \hat{a}_1(n) &= a_1(n) + a_3(n) \quad (n > 0), \\ \hat{a}_2(n) &= 2a_2(n) \quad (n < 0), \\ \hat{a}_3(n) &= 3a_1(n) - a_3(n) \quad (n > 0). \end{aligned}$$

Triples $(\mathcal{O}, \mathfrak{a}, \delta)$

We consider triples $(\mathcal{O}, \mathfrak{a}, \delta)$, where $\mathcal{O}$ is a nondegenerate cubic ring, $\mathfrak{a}$ is a fractional ideal of $\mathcal{O}$ and δ is an invertible element of $k = \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $\mathfrak{a}^2 \subset \delta\mathcal{O}$ and $N_{k/\mathbb{Q}}(\delta) = N_{\mathcal{O}}(\mathfrak{a})^2$. Here $N_{\mathcal{O}}(\mathfrak{a})$ is the norm of $\mathfrak{a}$ as a fractional $\mathcal{O}$-ideal, i. e. $N_{\mathcal{O}}(\mathfrak{a}) = (\mathcal{O} : \mathfrak{a})$ for $\mathfrak{a} \subset \mathcal{O}$. Two such triples $(\mathcal{O}, \mathfrak{a}, \delta)$ and $(\mathcal{O}', \mathfrak{a}', \delta')$ are called *equivalent* if there exists an isomorphism $\phi : \mathcal{O} \to \mathcal{O}'$ and $\kappa \in \mathcal{O}' \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $\mathfrak{a}' = \kappa\phi(\mathfrak{a})$, $\delta' = \kappa^2\phi(\delta)$. M. Bhargava proved the following theorem (Higher composition laws II, *Ann. of Math*. **159** (2004), 865–886).

Thorem 2.1 (Bhargava)

There is a canonical bijection between the set of nondegenerate Γ-orbits on $\hat{L}$ and the set of equivalence classes of triples $(\mathcal{O}, \mathfrak{a}, \delta)$. Under this bijection, the discriminant of a pair of integral matrices of degree three equals the discriminant of the corresponding cubic ring.

Bijection of Theorem 2.1

Let $\mathcal{O}$ be a nondegenerate cubic ring and having $\mathbb{Z}$-basis $\{1, \omega, \theta\}$ and multiplication table (1.1) with $a, b, c, d \in \mathbb{Z}$. Let $\mathfrak{a}$ be a fractional $\mathcal{O}$-ideal and δ be an invertible element of $k = \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $\mathfrak{a}^2 \subset \delta\mathcal{O}$ and $N_{k/\mathbb{Q}}(\delta) = N_{\mathcal{O}}(\mathfrak{a})^2$. We take a $\mathbb{Z}$-basis $\{\alpha_1, \alpha_2, \alpha_3\}$ of the ideal $\mathfrak{a}$ having the same orientation as $\{1, \omega, \theta\}$. Since $\mathfrak{a}^2 \subset \delta\mathcal{O}$, there exist integers a_{ij}, b_{ij} and c_{ij} such that

(2.1) $$\alpha_i\alpha_j = \delta(c_{ij} + b_{ij}\omega + a_{ij}\theta).$$

We put $A = (a_{ij})$, $B = (b_{ij})$. Then we have

$$\hat{F}_{(A,B)}(u) = au_1^3 + bu_1^2u_2 + cu_1u_2^2 + du_2^3.$$

The correspondence $(\mathcal{O}, \mathfrak{a}, \delta) \mapsto \Gamma \cdot (A, B)$ gives the bijection of Theorem 2.1.

Isotropy group $\Gamma_{(A,B)}$

We denote by $\Gamma_{(A,B)}$ the isotropy group in Γ of a nondegenerate pair $(A,B) \in \hat{L}$.

Corollary 2.2 (Bhargava)

For any nondegenerate pair $(A,B) \in \hat{L}$, there exists a homomorphism $\Gamma_{(A,B)} \to \mathrm{Aut}(\mathcal{O})$ with kernel isomorphic to $U_2^+(\mathcal{O}_0)$. Here $(\mathcal{O}, \mathfrak{a})$ is the pair corresponding to (A,B) as in Theorem 2.1, $\mathcal{O}_0 = \mathrm{End}_{\mathcal{O}}(\mathfrak{a})$ is the endomorphism ring of $\mathfrak{a}$, and $U_2^+(\mathcal{O}_0)$ denotes the group of units of $\mathcal{O}_0$ having order dividing 2 and positive norm.

Remark 2.3

The original statement of Corollary 2.2 is $\Gamma_{(A,B)} \cong U_2^+(\mathcal{O}_0) \rtimes \mathrm{Aut}(\mathcal{O})$, which is not correct. So we have given a weaker statement than the original one.

$\hat{L}(\mathcal{O}, \mathcal{O}_0)$

Let k be an étale cubic algebra over $\mathbb{Q}$ and let $\mathcal{O}$ be an order of k with square free index $f = (\mathcal{O}_k : \mathcal{O})$. By a theorem of elementary divisors, there exists a basis $\{1, \omega, \theta\}$ of $\mathcal{O}_k$ with $\omega, \theta \in k^\times$ such that $\{1, f\omega, \theta\}$ is a basis of $\mathcal{O}$ and the multiplication in $\mathcal{O}_k$ is given by (1.1). We note that f divides d.

Let $(\mathcal{O}, \mathfrak{a}, \delta)$ be a triplet corresponding to $(A, B) \in \hat{L}(\mathcal{O})$ and put $\mathcal{O}_0 = \mathrm{End}_{\mathcal{O}}(\mathfrak{a})$, then $\mathcal{O} \subset \mathcal{O}_0 \subset \mathcal{O}_k$. We set

$$\hat{L}(\mathcal{O}) = \{(A, B) \in \hat{L} \,:\, R(\hat{F}_{(A,B)}) \cong \mathcal{O}\},$$

$$\hat{L}(\mathcal{O}, \mathcal{O}_0) = \{(A, B) \in \hat{L}(\mathcal{O}) \,:\, \mathrm{End}_{\mathcal{O}}(\mathfrak{a}) = \mathcal{O}_0\},$$

$$\hat{L}(\mathcal{O}) = \bigcup_{\mathcal{O} \subset \mathcal{O}_0 \subset \mathcal{O}_k} \hat{L}(\mathcal{O}, \mathcal{O}_0).$$

Γ_1-orbits and Γ-orbits

We put

$$\hat{L}(F, \mathcal{O}_0) = \{(A, B) \in \hat{L}(\mathcal{O}, \mathcal{O}_0) \, : \, \hat{F}_{(A,B)} = F\}.$$

For any $(A, B) \in \hat{L}(\mathcal{O}, \mathcal{O}_0)$, there exists an element $\gamma_2 \in \Gamma_2$ such that $\hat{F}_{\gamma_2(A,B)} = F$. Hence we have a natural surjective mapping $\Gamma_1 \backslash \hat{L}(F, \mathcal{O}_0) \to \Gamma \backslash \hat{L}(\mathcal{O}, \mathcal{O}_0)$. By Corollary 2.2, we have

Lemma 2.4

The cardinality of the inverse image of $\Gamma \cdot (A, B)$ *is* $|\operatorname{Aut}(\mathcal{O})| \cdot |U_2^+(\mathcal{O}_0)|/|\Gamma_{(A,B)}|$. *Hence*

$$|\Gamma_1 \backslash \hat{L}(F, \mathcal{O}_0)| = \sum_{(A,B) \in \Gamma \backslash \hat{L}(\mathcal{O}, \mathcal{O}_0)} \frac{|\operatorname{Aut}(\mathcal{O})| \cdot |U_2^+(\mathcal{O}_0)|}{|\Gamma_{(A,B)}|}.$$

Ideal $\mathfrak{j}(\mathcal{O}, \mathcal{O}_0)$

Let $(\mathcal{O}, \mathfrak{a}, \delta)$ be a triplet and put $\mathcal{O}_0 = \mathrm{End}_{\mathcal{O}}(\mathfrak{a})$. Since $\mathcal{O} \subset \mathcal{O}_0 \subset \mathcal{O}_k$ and $\mathcal{O}_k/\mathcal{O} \cong \mathbb{Z}/f\mathbb{Z}$, $\mathcal{O}_0 = [1, g\omega, \theta]$ for some positive divisor g of f. We write $f = gh$.
The conductor $\mathfrak{f}$ of $\mathcal{O}$ (the maximal $\mathcal{O}_k$-ideal contained in $\mathcal{O}$) is given by $\mathfrak{f} = [f, f\omega, \theta]$. Similarly the conductor $\mathfrak{g}$ of $\mathcal{O}_0$ is given by $\mathfrak{g} = [g, g\omega, \theta]$. We note that $\mathcal{O} = \mathbb{Z} + \mathfrak{f}$, $\mathcal{O}_0 = \mathbb{Z} + \mathfrak{g}$.
We put $\mathfrak{j} = [h, f\omega, \theta]$. Then $\mathfrak{j} \subset \mathcal{O}$ and $\mathfrak{j}$ is an $\mathcal{O}_0$-ideal. Since $f = gh$ is square free, g and h are coprime to each other. Hence $\mathfrak{j} + \mathfrak{g} = \mathcal{O}_0$. We put $\mathfrak{h} = [h, h\omega, \theta]$ and $\mathfrak{h}' = [h, \omega, \theta + c]$. Then $\mathfrak{h}$ and $\mathfrak{h}'$ are $\mathcal{O}_k$-ideals such that $\mathfrak{h}\mathfrak{h}' = h\mathcal{O}_k$. So $\mathfrak{h}$ is an invertible $\mathcal{O}_k$-ideal. Further we have $\mathfrak{j}\mathcal{O}_k = \mathfrak{h}$. This implies that $\mathfrak{j}$ is an invertible $\mathcal{O}_0$-ideal. It is easy to see that $\mathfrak{j}$ is the largest $\mathcal{O}_0$-ideal contained in $\mathcal{O}$. So we write $\mathfrak{j} = \mathfrak{j}(\mathcal{O}, \mathcal{O}_0)$.

Fröhlich's result

Let R be a nondegenerate ring of rank n. We say that R is *weakly self dual* if every proper R-ideal $\mathfrak{a}$ ($\mathrm{End}_R(\mathfrak{a}) = R$) is an invertible R-ideal. Since the index $g = (\mathcal{O}_k : \mathcal{O}_0)$ is square free, the ring $\mathcal{O}_0$ is weakly self dual by a result of Fröhlich (Invariants for modules over commutative separable orders, Quart. J. Math. Oxford (2) **16** (1965), 193–232.). Hence each proper $\mathcal{O}_0$-ideal is an invertible $\mathcal{O}_0$-ideal. In particular $\mathfrak{a}$ is an invertible $\mathcal{O}_0$-ideal.

$$N_{\mathcal{O}_0}(\mathfrak{a}) = (\mathcal{O}_0 : \mathcal{O})N_{\mathcal{O}}(\mathfrak{a}) = hN_{\mathcal{O}}(\mathfrak{a}).$$

By a result of Fröhlich, a proper $\mathcal{O}_0$-ideal $\mathfrak{b}$ is invertible if and only if $(\mathcal{O}_k\mathfrak{b} : \mathfrak{b}) = (\mathcal{O}_k : \mathcal{O}_0)$. Hence for any proper integral $\mathcal{O}_0$-ideal $\mathfrak{b}$, we have

$$N_{\mathcal{O}_0}(\mathfrak{b}) = (\mathcal{O}_0 : \mathfrak{b}) = \frac{(\mathcal{O}_k : \mathcal{O}_k\mathfrak{b})(\mathcal{O}_k\mathfrak{b} : \mathfrak{b})}{(\mathcal{O}_k : \mathcal{O}_0)} = (\mathcal{O}_k : \mathcal{O}_k\mathfrak{b}) = N(\mathcal{O}_k\mathfrak{b}).$$

$\delta^{-1}\mathfrak{a}^2 = \mathfrak{j}(\mathcal{O}, \mathcal{O}_0)$

This implies that the norm $N_{\mathcal{O}_0}$ is multiplicative for proper $\mathcal{O}_0$-ideals. In particular, we have

$$N_{\mathcal{O}_0}(\delta^{-1}\mathfrak{a}^2) = N_{k/\mathbb{Q}}(\delta)^{-1}N_{\mathcal{O}_0}(\mathfrak{a})^2 = N_{k/\mathbb{Q}}(\delta)^{-1}h^2N_{\mathcal{O}}(\mathfrak{a})^2 = h^2.$$

Since $\mathfrak{j} = [h, gh\omega, \theta]$ and $\mathcal{O}_0 = [1, g\omega, \theta]$, we have $N_{\mathcal{O}_0}(\mathfrak{j}) = h^2$. Then the inclusion $\delta^{-1}\mathfrak{a}^2 \subset \mathfrak{j}$ implies $\delta^{-1}\mathfrak{a}^2 = \mathfrak{j}$.
Conversely, if $\mathfrak{a}$ is an invertible $\mathcal{O}_0$-ideal and $\delta \in k^\times$ has positive norm such that $\delta^{-1}\mathfrak{a}^2 = \mathfrak{j}$, then the triplet $(\mathcal{O}, \mathfrak{a}, \delta)$ is as in Theorem 2.1.
We denote by $I_{\mathcal{O}_0}$ the group of invertible fractional $\mathcal{O}_0$-ideals. We put

$$\mathscr{I}(\mathcal{O}, \mathcal{O}_0) = \{(\mathfrak{a}, \delta) \in I_{\mathcal{O}_0} \times k^\times \,:\, \delta^{-1}\mathfrak{a}^2 = \mathfrak{j}(\mathcal{O}, \mathcal{O}_0),\ N_{k/\mathbb{Q}}(\delta) > 0\}.$$

We say that two elements $(\mathfrak{a}, \delta)$ and $(\mathfrak{a}', \delta')$ in $\mathscr{I}(\mathcal{O}, \mathcal{O}_0)$ are *equivalent* if there exists an element $\kappa \in k^\times$ such that $\mathfrak{a}' = \kappa\mathfrak{a}$ and $\delta' = \kappa^2\delta$.

For any $(\mathfrak{a}, \delta) \in \mathscr{I}(\mathcal{O}, \mathcal{O}_0)$, the correspondence $(\mathfrak{a}, \delta) \mapsto \Gamma_1 \cdot (A, B)$ induces a bijection $\mathscr{I}(\mathcal{O}, \mathcal{O}_0)/\sim \rightarrow \Gamma_1 \backslash \hat{L}(F, \mathcal{O}_0)$, where the equivalence class of $(\mathcal{O}, \mathfrak{a}, \delta)$ corresponds to $\Gamma \cdot (A, B)$ in Theorem 2.1.

We denote by $X(\mathcal{O}, \mathcal{O}_0)$ the subgroup of $\mathrm{Cl}_{\mathcal{O}_0} / \mathrm{Cl}^2_{\mathcal{O}_0}$ generated by the ideal class of $\mathfrak{j}(\mathcal{O}, \mathcal{O}_0)$. By the definition of $\mathscr{I}(\mathcal{O}, \mathcal{O}_0)$, $\mathscr{I}(\mathcal{O}, \mathcal{O}_0) \neq \emptyset$ if and only if $X(\mathcal{O}, \mathcal{O}_0)$ is trivial. We denote by $\mathrm{Cl}^{(2)}_{\mathcal{O}_0}$ the two torsion subgroup of $\mathrm{Cl}_{\mathcal{O}_0}$. We have the following equation

$$|\mathscr{I}(\mathcal{O}, \mathcal{O}_0)/\sim| = |\,\mathrm{Cl}^{(2)}_{\mathcal{O}_0}|(U^+(\mathcal{O}_0) : U^+(\mathcal{O}_0)^2)\,(2 - |X(\mathcal{O}, \mathcal{O}_0)|).$$

By Lemma 2.4, we have

$$\sum_{(A,B) \in \Gamma \backslash \hat{L}(\mathcal{O}, \mathcal{O}_0)} \frac{1}{|\Gamma_{(A,B)}|} = \frac{(U^+(\mathcal{O}_0) : U^+(\mathcal{O}_0)^2)}{|\,\mathrm{Aut}(\mathcal{O})|\,|U_2^+(\mathcal{O}_0)|}\,|\,\mathrm{Cl}^{(2)}_{\mathcal{O}_0}| \times (2 - |X(\mathcal{O}, \mathcal{O}_0)|).$$

If $\mathrm{Disc}(k) > 0$, then $\hat{L}(\mathcal{O}, \mathcal{O}_0)$ is decomposed into two subsets $\hat{L}_1(\mathcal{O}, \mathcal{O}_0)$ and $\hat{L}_3(\mathcal{O}, \mathcal{O}_0)$. Using the narrow ideal class group $\mathrm{Cl}_{\mathcal{O}_0,+}$, we obtain

$$\sum_{(A,B)\in\Gamma\backslash\hat{L}_1(\mathcal{O},\mathcal{O}_0)} \frac{1}{|\Gamma_{(A,B)}|} = \frac{(U_+(\mathcal{O}_0) : U_+(\mathcal{O}_0)^2)}{2^3|\,\mathrm{Aut}(\mathcal{O})|\,|U_2(\mathcal{O}_0)|}\,|\,\mathrm{Cl}^{(2)}_{\mathcal{O}_0,+}\,| \times (2 - |X_+(\mathcal{O}, \mathcal{O}_0)|).$$

Here $U_+(\mathcal{O}_0)$ denotes the group of totally positive units in $\mathcal{O}_0$ and $U_2(\mathcal{O}_0)$ denotes the group of units in $\mathcal{O}_0$ having order dividing 2. $X_+(\mathcal{O}, \mathcal{O}_0)$ is the subgroup of $\mathrm{Cl}_{\mathcal{O}_0,+} / \mathrm{Cl}^2_{\mathcal{O}_0,+}$ generated by the ideal class of $\mathrm{j}(\mathcal{O}, \mathcal{O}_0)$.

Proposition 2.5

If $\mathrm{Disc}(k) > 0$, *then*

$$\begin{aligned}
&\sum\nolimits_{y\in\Gamma\backslash\hat{L}_1(\mathcal{O},\mathcal{O}_0)} \mu(y) \\
&= \frac{(U_+(\mathcal{O}_0) : U_+(\mathcal{O}_0)^2)\cdot|U_2(\mathcal{O}_0)|}{2^3|\,\mathrm{Aut}(\mathcal{O})|\cdot|U_2^+(\mathcal{O}_0)|}\,|\,\mathrm{Cl}^{(2)}_{\mathcal{O}_0,+}\,|\,(2-|X_+(\mathcal{O},\mathcal{O}_0)|), \\
&\sum\nolimits_{y\in\Gamma\backslash\hat{L}_1(\mathcal{O},\mathcal{O}_0)} \mu(y) + \sum\nolimits_{y\in\Gamma\backslash\hat{L}_3(\mathcal{O},\mathcal{O}_0)} \mu(y) \\
&= \frac{(U^+(\mathcal{O}_0) : U^+(\mathcal{O}_0)^2)}{|\,\mathrm{Aut}(\mathcal{O})|\cdot|U_2^+(\mathcal{O}_0)|}\,|\,\mathrm{Cl}^{(2)}_{\mathcal{O}_0}\,|\,(2-|X(\mathcal{O},\mathcal{O}_0)|),
\end{aligned}$$

otherwise

$$\sum_{y\in\Gamma\backslash\hat{L}_2(\mathcal{O},\mathcal{O}_0)} \mu(y) = \frac{(U^+(\mathcal{O}_0) : U^+(\mathcal{O}_0)^2)}{|\,\mathrm{Aut}(\mathcal{O})|\cdot|U_2^+(\mathcal{O}_0)|}\,|\,\mathrm{Cl}^{(2)}_{\mathcal{O}_0}\,|\,(2-|X(\mathcal{O},\mathcal{O}_0)|).$$

SL_2-invariants of pairs of ternary quadratic forms

Let (A, B) be a pair of integral ternary quadratic forms and write

$$A(v) = \sum_{1 \le i \le j \le 3} a_{ij} v_i v_j, \quad B(v) = \sum_{1 \le i \le j \le 3} b_{ij} v_i v_j.$$

We put $a_{ji} = a_{ij}$, $b_{ji} = b_{ij}$ and define $\lambda^{ij}_{k\ell} = \lambda^{ij}_{k\ell}(A, B)$ by

(3.1) $$\lambda^{ij}_{k\ell}(A, B) = \begin{vmatrix} a_{ij} & b_{ij} \\ a_{k\ell} & b_{k\ell} \end{vmatrix}.$$

For any permutation (i, j, k) of $(1, 2, 3)$, we put

$$c^i_{ii} = \pm \lambda^{ik}_{ij} + C_i, \qquad c^j_{ii} = \pm \lambda^{ii}_{ik},$$
$$c^i_{ij} = \pm (1/2) \lambda^{ik}_{jj} + (1/2) C_j, \quad c^k_{ij} = \pm \lambda^{jj}_{ii}.$$

Here $\pm$ denotes the signature of the permutation (i, j, k) and $C_1 = \lambda^{23}_{11}$, $C_2 = -\lambda^{13}_{22}$, $C_3 = \lambda^{12}_{33}$. Then $c^k_{ij} \in \mathbb{Z}$ for $k > 0$. We put

(3.2) $$c^0_{ij} = \sum_{r=1}^{3} (c^r_{jk} c^k_{ri} - c^r_{ij} c^k_{rk}).$$

Quartic rings and pairs of ternary quadratic forms

Let $Q(A,B)$ be a free $\mathbb{Z}$-module with basis $\{\alpha_0 = 1, \alpha_1, \alpha_2, \alpha_3\}$ and the multiplication of $Q(A,B)$ is given by

$$(3.3) \qquad \alpha_i \alpha_j = \sum_{k=0}^{3} c_{ij}^k \alpha_k \quad (i,j \in \{1,2,3\}).$$

Then $Q(A,B)$ becomes a quartic ring. We defined an integral binary cubic form $F_{(A,B)}(u)$ in $u = (u_1, u_2)$ by $4\det(u_1 A - u_2 B)$. The discriminant of $Q(A,B)$ is equal to $\mathrm{Disc}(A,B) = \mathrm{Disc}(F_{(A,B)})$. We put $R(A,B) = R(F_{(A,B)})$. Then the discriminant of $R(A,B)$ is also equal to $\mathrm{Disc}(A,B)$.

For any quartic ring Q and for an element $x \in Q$, we denote by x, x', x'', x''' the 'conjugates' of x. We put $\phi(x) = xx' + x''x'''$. Then all $\phi(x)$ are contained in some cubic ring $R^{\mathrm{inv}}(Q)$. A *cubic resolvent ring* of Q is a cubic ring R such that $\mathrm{Disc}(R) = \mathrm{Disc}(Q)$ and $R \supset R^{\mathrm{inv}}(Q)$.

HCL III

Bhargava proved the following theorem (Higher composition laws III, *Ann. of Math*. **159** (2004), 1329–1360.).

Thorem 3.1 (Bhargava)

The correspondence $(A, B) \mapsto (Q(A, B), R(A, B))$ *induces a canonical bijection between the set of* Γ*-orbits of nondegenerate pairs of integral ternary quadratic forms and the set of isomorphism classes of pairs* (Q, R)*, where* Q *is a nondegenerate quartic ring and* R *is a cubic resolvent ring of* Q*.*

Corollary 3.2 (Bhargava)

Every quartic ring has a cubic resolvent ring. A primitive quartic ring has a unique cubic resolvent ring up to isomorphism. In particular, every maximal quartic ring has a unique cubic resolvent ring.

$\Gamma_{(A,B)}$ and $\mathrm{Aut}(Q(A,B))$

We call (A,B) *primitive* if $\gcd(\lambda^{ij}_{k\ell}(A,B)) = 1$.

Proposition 3.3

For any nondegenerate pair (A,B) of integral ternary quadratic forms, there exists an injective group homomorphism $\Gamma_{(A,B)} \to \mathrm{Aut}(Q(A,B))$. Further if (A,B) is primitive, then the homomorphism is an isomorphism.

Hence we have $\mu(A,B) = 1/|\Gamma_{(A,B)}| = 1/|\mathrm{Aut}(Q(A,B))|$ if (A,B) is primitive.

Relative discriminant $\mathrm{Disc}(k_6/k)$

Let k be a non-Galois cubic field and $\mathcal{O}$ be an order of k such that the index $f = (\mathcal{O}_k : \mathcal{O})$ is odd and square free. Let K be an S_4-quartic field and assume that an order Q of K has a cubic resolvent ring isomorphic to $\mathcal{O}$. We denote by $\tilde{K}$ the Galois closure of K. Let k_6 be the non-Galois sextic field such that $k \subset k_6 \subset \tilde{K}$ and $k_6 = k(\sqrt{\alpha})$ for some $\alpha \in k^\times$ with $N_{k/\mathbb{Q}}(\alpha) = a^2$, $a \in \mathbb{Q}^\times$. Then the norm of the relative discriminant $N(\mathrm{Disc}(k_6/k)) = g^2$ for some $g \in \mathbb{N}$ and $\mathrm{Disc}(K) = g^2 \,\mathrm{Disc}(k)$. We put $h = (\mathcal{O}_K : Q)$. Then $\mathrm{Disc}(Q) = \mathrm{Disc}(\mathcal{O}) = f^2 \,\mathrm{Disc}(k)$ implies $f = gh$.
We denote by $\mathfrak{f}$ the conductor of $\mathcal{O}$. Then $N(\mathfrak{f}) = f^2$. For each prime divisor p of f, we denote by $\mathfrak{f}_p$ the p-part of $\mathfrak{f}$ and put $\mathfrak{g} = \prod_{p|g} \mathfrak{f}_p$ and $\mathfrak{h} = \prod_{p|h} \mathfrak{f}_p$. We can prove that $\mathfrak{g} = \mathrm{Disc}(k_6/k)$.
Moreover the conductor of the unique cubic resolvent ring of $\mathcal{O}_K$ is $\mathfrak{g}$.

Quartic rings Q having cubic resolvent ring $\mathcal{O}$

We denote by $a_K(\mathfrak{h})$ the number of quartic rings Q contained in K having cubic resolvent ring isomorphic to $\mathcal{O}$. We denote by H the subgroup of $I_k(\mathfrak{g})$ corresponding to the quadratic extension k_6/k by class field theory and χ the character of $I_k(\mathfrak{g})$ such that $\ker\chi = H$. Here $I_k(\mathfrak{g})$ is the group of fractional ideals of k which are relatively prime to $\mathfrak{g}$. We can prove the following formula.

(4.1)
$$a_K(\mathfrak{h}) = \prod_{p|h}(1+\chi(\mathfrak{f}_p)).$$

If $\mathrm{Disc}(k) < 0$, then the number of quartic rings Q with fixed cubic resolvent ring $\mathcal{O}$ is given by the sum

$$(4.2) \qquad \sum_{g|f} \sum_{K \in \mathcal{K}_k(\mathfrak{g})} a_K(\mathfrak{h}) = \sum_{g|f} \sum_{K \in \mathcal{K}_k(\mathfrak{g})} \prod_{p|h} (1 + \chi_K(\mathfrak{f}_p)),$$

where we denote by $\mathcal{K}_k(\mathfrak{g})$ the set of isomorphism classes of quartic fields K satisfying the following conditions:

(a) The normal closure $\tilde{K}$ of K over $\mathbb{Q}$ has Galois group S_4 and contains k.

(b) The unique cubic resolvent ring of the maximal order $\mathcal{O}_K$ is isomorphic to $\mathcal{O}_0 = \mathbb{Z} + \mathfrak{g}$.

(c) K is totally real if $\mathrm{Disc}(k) > 0$.

If $\mathrm{Disc}(k) > 0$, then the sum (4.2) gives the number of such quartic rings contained in some totally real S_4-quartic fields.

If $\mathrm{Disc}(k) > 0$, we denote by $\mathcal{K}_k(\mathfrak{g}\mathfrak{f}_\infty)$ the set of isomorphism classes of quartic fields K satisfying the conditions (a) and (b) above (including totally imaginary fields). If $\mathrm{Disc}(k) > 0$, then the number of quartic rings contained in some quartic fields with fixed cubic resolvent ring $\mathcal{O}$ is given by

$$\text{(4.3)} \qquad \sum_{g|f} \sum_{K \in \mathcal{K}_k(\mathfrak{g}\mathfrak{f}_\infty)} a_K(\mathfrak{h}) = \sum_{g|f} \sum_{K \in \mathcal{K}_k(\mathfrak{g}\mathfrak{f}_\infty)} \prod_{p|h} (1 + \chi_K(\mathfrak{f}_p)).$$

By class field theory and quadratic reciprocity laws over the cubic field k, we obtain the following formulae.

$$\text{(4.4)} \qquad \begin{aligned} \sum_{\mathfrak{g}|\mathfrak{f}} |\mathcal{K}_k(\mathfrak{g})| &= |\,\mathrm{Cl}_{\mathcal{O}}^{(2)}\,| - 1, \\ \sum_{\mathfrak{g}|\mathfrak{f}} |\mathcal{K}_k(\mathfrak{g}\mathfrak{f}_\infty)| &= |\,\mathrm{Cl}_{\mathcal{O},+}^{(2)}\,| - 1. \end{aligned}$$

We can rewrite the right hand sides of (4.2) and (4.3) so that we finally obtain the following formulae:

$$(4.5) \quad \sum_{g|f} \sum_{K \in \mathcal{K}_k(\mathfrak{g})} a_K(\mathfrak{h}) = \sum_{g|f} |\operatorname{Cl}^{(2)}_{R_g}|(2 - |X(\mathcal{O}, R_g)|) - 2^{\omega(f)},$$

$$(4.6) \quad \sum_{g|f} \sum_{K \in \mathcal{K}_k(\mathfrak{g}\mathfrak{f}_\infty)} a_K(\mathfrak{h}) = \sum_{g|f} |\operatorname{Cl}^{(2)}_{R_g,+}|(2 - |X_+(\mathcal{O}, R_g)|) - 2^{\omega(f)}$$

$$(\operatorname{Disc}(k) > 0).$$

Here $\omega(f)$ is the number of distinct prime divisors of f and $R_g = \mathbb{Z} + \mathfrak{g}$.
We see that $2^{\omega(f)}$ equals the number of quartic rings contained in the reducible algebra $\mathbb{Q} \oplus k$ having cubic resolvent ring $\mathcal{O}$.

We write $x = (A, B) \in L(\mathcal{O})$, $\mu(x) = 1/|\Gamma_x|$. By Proposition 3.3, $\mu(x) = 1/|\operatorname{Aut}(Q(A, B))| = 1$. It follows from Theorem 3.1, Corollary 3.2, (4.5) and (4.6) that if $\operatorname{Disc}(k) > 0$, then

(4.7)

$$\sum_{x \in \Gamma \backslash L_1(\mathcal{O})} \mu(x) = \sum_{g|f} |\operatorname{Cl}_{R_g}^{(2)}|(2 - |X(\mathcal{O}, R_g)|),$$

$$\sum_{x \in \Gamma \backslash L_1(\mathcal{O})} \mu(x) + \sum_{x \in \Gamma \backslash L_3(\mathcal{O})} \mu(x) = \sum_{g|f} |\operatorname{Cl}_{R_g,+}^{(2)}|(2 - |X_+(\mathcal{O}, R_g)|),$$

otherwise

(4.8)
$$\sum_{x \in \Gamma \backslash L_2(\mathcal{O})} \mu(x) = \sum_{g|f} |\operatorname{Cl}_{R_g}^{(2)}|(2 - |X(\mathcal{O}, R_g)|).$$

So we finally complete the proof of Theorem 1 for a non-Galois cubic field k by Proposition 2.5, (4.7) and (4.8).