3元2次形式のペアのゼータ関数に関する予想

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§1　V: 3 元 2 次形式のペアのなすベクトル空間 $/\mathbb{C}$,
$v = (v_1, v_2, v_3)$ を 3 変数，$x = (x_1, x_2) \in V$,

$$x_k(v) = \sum_{1 \le i \le j \le 3} x_{k,ij} v_i v_j \qquad (k = 1, 2).$$

$$x_k \longleftrightarrow \begin{pmatrix} x_{k,11} & x_{k,12}/2 & x_{k,13}/2 \\ x_{k,12}/2 & x_{k,22} & x_{k,23}/2 \\ x_{k,13}/2 & x_{k,23}/2 & x_{k,33} \end{pmatrix}.$$

$u = (u_1, u_2)$ を 2 変数,

$$\begin{aligned} F_x(u) &= 4\det(u_1 x_1 - u_2 x_2) = a u_1^3 + b u_1^2 u_2 + c u_1 u_2^2 + d u_2^3, \\ \mathrm{Disc}(x) &= \mathrm{Disc}(F_x) = 18abcd + b^2c^2 - 4ac^3 - 4b^3 d - 27a^2 d^2, \\ S &= \{x \in V : \mathrm{Disc}(x) = 0\}. \end{aligned}$$

$x \in V \setminus S$ に対して

$$\begin{aligned} \mathrm{Zero}(x) &= \{v \in \mathbb{P}^2(\mathbb{C}) \mid x_1(v) = x_2(v) = 0\}, \\ V_1 &= \{x \in V_{\mathbb{R}} \setminus S_{\mathbb{R}} : |\,\mathrm{Zero}(x) \cap \mathbb{P}^2(\mathbb{R})| = 4\}, \\ V_2 &= \{x \in V_{\mathbb{R}} \setminus S_{\mathbb{R}} : |\,\mathrm{Zero}(x) \cap \mathbb{P}^2(\mathbb{R})| = 2\}, \\ V_3 &= \{x \in V_{\mathbb{R}} \setminus S_{\mathbb{R}} : |\,\mathrm{Zero}(x) \cap \mathbb{P}^2(\mathbb{R})| = 0\}. \end{aligned}$$

$L = \{x = (x_1, x_2) : x_{k,ij} \in \mathbb{Z}\} \subset V_{\mathbb{R}}$,　$\hat{L}$: $\mathbb{Z}$ 係数 3 次対称行列のペア全体

$y \in \hat{L}$, $\hat{F}_y(u) = \det(u_1 y_1 - u_2 y_2)$, $\mathrm{Disc}^*(y) = \mathrm{Disc}(\hat{F}_y) = 2^{-8}\,\mathrm{Disc}(y)$.
$x \in L$, $\Gamma_x = \{\gamma \in \Gamma : \gamma x = x\}$, $\mu(x) = 1/|\Gamma_x|$.
$i = 1, 2, 3$ と $n \in \mathbb{Z}$, $n \ne 0$ に対して

$$a_i(n) = \sum_{\substack{x \in \Gamma \backslash (L \cap V_i) \\ \mathrm{Disc}(x) = n}} \mu(x), \qquad \hat{a}_i(n) = \sum_{\substack{y \in \Gamma \backslash (\hat{L} \cap V_i) \\ \mathrm{Disc}^*(y) = n}} \mu(y).$$

ゼータ関数の定義 $(i = 1, 2, 3)$

$$\xi_i(L, s) = \sum_{x \in \Gamma \backslash L \cap V_i} \frac{\mu(x)}{|\,\mathrm{Disc}(x)|^s} = \sum_{n=1}^{\infty} \frac{a_i((-1)^{i-1} n)}{n^s},$$

$$\xi_i(\hat{L}, s) = \sum_{y \in \Gamma \backslash \hat{L} \cap V_i} \frac{\mu(y)}{|\,\mathrm{Disc}^*(y)|^s} = \sum_{n=1}^{\infty} \frac{\hat{a}_i((-1)^{i-1} n)}{n^s}.$$

関数等式

$$\begin{pmatrix} \xi_1(L,1-s) \\ \xi_2(L,1-s) \\ \xi_3(L,1-s) \end{pmatrix} = \Gamma(s)^4 \Gamma\left(s-\frac{1}{6}\right)^2 \Gamma\left(s+\frac{1}{6}\right)^2 \Gamma\left(s-\frac{1}{4}\right)^2 \Gamma\left(s+\frac{1}{4}\right)^2$$
$$\times 2^{8s} 3^{6s} \pi^{-12s} \big(u_{ji}^*(s)\big) \begin{pmatrix} \xi_1(\hat{L},s) \\ \xi_2(\hat{L},s) \\ \xi_3(\hat{L},s) \end{pmatrix}.$$

$u_{ij}^*(s)$ は $q=\exp(\pi\sqrt{-1}s)$ と q^{-1} の高々 6 次の多項式.

予想.

$$\begin{aligned} \xi_1(\hat{L},s) &= \xi_1(L,s)+\xi_3(L,s), \\ \xi_2(\hat{L},s) &= 2\xi_2(L,s), \\ \xi_3(\hat{L},s) &= 3\xi_1(L,s)-\xi_3(L,s). \end{aligned}$$

3 次環 $R(F)$ 　$\mathbb{Z}$ 係数 2 元 3 次形式

$$F(u) = au_1^3 + bu_1^2u_2 + cu_1u_2^2 + du_2^3, \quad a,b,c,d \in \mathbb{Z}$$

に対して $\{1,\omega,\theta\}$ を基底とする $\mathbb{Z}$ 加群に乗法を積を次のように定義したもの.

$$\begin{aligned} \omega^2 &= -ac + b\omega - a\theta, \\ \theta^2 &= -bd + d\omega - c\theta, \\ \omega\theta &= -ad. \end{aligned} \tag{1.1}$$

3 次環 $\mathcal{O}$ に対して

$$\begin{aligned} L(\mathcal{O}) &= \{x \in L : R(F_x) \cong \mathcal{O}\}, & \hat{L}(\mathcal{O}) &= \{y \in \hat{L} : R(\hat{F}_y) \cong \mathcal{O}\}, \\ L_i(\mathcal{O}) &= L(\mathcal{O}) \cap V_i \quad (i=1,2,3), & \hat{L}_i(\mathcal{O}) &= \hat{L}(\mathcal{O}) \cap V_i \quad (i=1,2,3). \end{aligned}$$

定理 1. k を 3 次体, $\mathcal{O}_k$ を k の極大整環, $\mathcal{O}$ を k の整環で指数 $(\mathcal{O}_k:\mathcal{O})$ は平方因数をもたないとする. このとき

$$\sum_{y\in\Gamma\backslash\hat{L}_1(\mathcal{O})} \mu(y) = \sum_{x\in\Gamma\backslash L_1(\mathcal{O})} \mu(x) + \sum_{x\in\Gamma\backslash L_3(\mathcal{O})} \mu(x) \quad (\mathrm{Disc}(k)>0),$$
$$\sum_{y\in\Gamma\backslash\hat{L}_2(\mathcal{O})} \mu(y) = 2\sum_{x\in\Gamma\backslash L_2(\mathcal{O})} \mu(x) \quad (\mathrm{Disc}(k)<0),$$
$$\sum_{y\in\Gamma\backslash\hat{L}_3(\mathcal{O})} \mu(y) = 3\sum_{x\in\Gamma\backslash L_1(\mathcal{O})} \mu(x) - \sum_{x\in\Gamma\backslash L_3(\mathcal{O})} \mu(x) \quad (\mathrm{Disc}(k)>0).$$

定理 2. n が基本判別式 (2 次体の判別式であるような整数) ならば次が成り立つ.

$$\begin{aligned}\hat{a}_1(n) &= a_1(n) + a_3(n) \quad (n > 0),\\ \hat{a}_2(n) &= 2a_2(n) \quad (n < 0),\\ \hat{a}_3(n) &= 3a_1(n) - a_3(n) \quad (n > 0).\end{aligned}$$

§ 2

k：エタール 3 次代数，$\mathcal{O}_k$: k の極大整環，
$\mathcal{O}$: 3 次環，$\mathrm{Disc}(\mathcal{O}) \neq 0$, $k = \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Q}$，$\mathfrak{a}$: $\mathcal{O}$ の分数イデアル，$\delta \in k^{\times}$，

$$\mathfrak{a}^2 \subset \delta\mathcal{O}, \qquad N_{k/\mathbb{Q}}(\delta) = N_{\mathcal{O}}(\mathfrak{a})^2$$

を満たす 3 つ組み $(\mathcal{O}, \mathfrak{a}, \delta)$ を考える．$\mathcal{O}_0 = \mathrm{End}_{\mathcal{O}}\,\mathfrak{a}$ とおく.

定理 2.1 (Bhargava)**.**

$$\Gamma \backslash \{(A, B) \in \hat{L} : \mathrm{Disc}^*(A, B) \neq 0\} \longleftrightarrow \{(\mathcal{O}, \mathfrak{a}, \delta)\}/\sim.$$

$\Gamma(A, B) \leftrightarrow (\mathcal{O}, \mathfrak{a}, \delta)$ のとき $\mathrm{Disc}^*(A, B) = \mathrm{Disc}(\mathcal{O})$.

3 次環 $\mathcal{O} \subset \mathcal{O}_k$, $f = (\mathcal{O}_k : \mathcal{O})$ は平方因数をもたない.
$\mathcal{O}_k = [1, \omega, \theta]$, $\mathcal{O} = [1, f\omega, \theta]$ (乗法は (1.1)).
$\mathcal{O} \subset \mathcal{O}_0 = [1, g\omega, \theta] \subset \mathcal{O}_k$, $f = gh$.
$\mathfrak{f} = [f, f\omega, \theta]$: $\mathcal{O}$ の導手，$\mathfrak{g} = [g, g\omega, \theta]$: $\mathcal{O}_0$ の導手，$\mathcal{O} = \mathbb{Z} + \mathfrak{f}$, $\mathcal{O}_0 = \mathbb{Z} + \mathfrak{g}$.
$\mathfrak{j} = \mathfrak{j}(\mathcal{O}, \mathcal{O}_0) = [h, f\omega, \theta]$: $\mathcal{O}$ に含まれる最大の $\mathcal{O}_0$ イデアル.

$I_{\mathcal{O}_0}$: 可逆分数 $\mathcal{O}_0$ イデアル全体のなす群
$\mathrm{Cl}_{\mathcal{O}_0}$: $\mathcal{O}_0$ のイデアル類群
$X(\mathcal{O}, \mathcal{O}_0) \subset \mathrm{Cl}_{\mathcal{O}_0} / \mathrm{Cl}_{\mathcal{O}_0}^2$: $\mathfrak{j}(\mathcal{O}, \mathcal{O}_0)$ のイデアル類で生成される部分群
$\mathrm{Cl}_{\mathcal{O}_0}^{(2)} = \{c \in \mathrm{Cl}_{\mathcal{O}_0} : c^2 = 1\}$
$U^+(\mathcal{O}_0)$: 正のノルムをもつ $\mathcal{O}_0$ の単数群
$U_+(\mathcal{O}_0)$: $\mathcal{O}_0$ の総正な単数群
$U_2(\mathcal{O}_0) = \{\varepsilon \in \mathcal{O}_0^{\times} : \varepsilon^2 = 1\}$
$X_+(\mathcal{O}, \mathcal{O}_0) \subset \mathrm{Cl}_{\mathcal{O}_0,+} / \mathrm{Cl}_{\mathcal{O}_0,+}^2$: $\mathfrak{j}(\mathcal{O}, \mathcal{O}_0)$ の狭義イデアル類で生成される部分群.

命題 2.6. $\mathrm{Disc}(k) > 0$ ならば

$$\begin{aligned}
&\sum\nolimits_{y\in\Gamma\backslash\hat{L}_1(\mathcal{O},\mathcal{O}_0)} \mu(y) \\
&= \frac{(U_+(\mathcal{O}_0) : U_+(\mathcal{O}_0)^2)\cdot|U_2(\mathcal{O}_0)|}{2^3|\operatorname{Aut}(\mathcal{O})|\cdot|U_2^+(\mathcal{O}_0)|}\,|\operatorname{Cl}^{(2)}_{\mathcal{O}_0,+}|\,(2-|X_+(\mathcal{O},\mathcal{O}_0)|),\\
&\sum\nolimits_{y\in\Gamma\backslash\hat{L}_1(\mathcal{O},\mathcal{O}_0)} \mu(y) + \sum\nolimits_{y\in\Gamma\backslash\hat{L}_3(\mathcal{O},\mathcal{O}_0)} \mu(y)\\
&= \frac{(U^+(\mathcal{O}_0) : U^+(\mathcal{O}_0)^2)}{|\operatorname{Aut}(\mathcal{O})|\cdot|U_2^+(\mathcal{O}_0)|}\,|\operatorname{Cl}^{(2)}_{\mathcal{O}_0}|\,(2-|X(\mathcal{O},\mathcal{O}_0)|),
\end{aligned}$$

$\mathrm{Disc}(k) < 0$ ならば

$$\sum_{y\in\Gamma\backslash\hat{L}_2(\mathcal{O},\mathcal{O}_0)} \mu(y) = \frac{(U^+(\mathcal{O}_0) : U^+(\mathcal{O}_0)^2)}{|\operatorname{Aut}(\mathcal{O})|\cdot|U_2^+(\mathcal{O}_0)|}\,|\operatorname{Cl}^{(2)}_{\mathcal{O}_0}|\,(2-|X(\mathcal{O},\mathcal{O}_0)|).$$

§ 3 $(A,B)\in L$ とし

$$A(v) = \sum\nolimits_{1\le i\le j\le 3} a_{ij}v_iv_j,\quad B(v) = \sum\nolimits_{1\le i\le j\le 3} b_{ij}v_iv_j.$$

$a_{ji} = a_{ij}$, $b_{ji} = b_{ij}$ とおく.
15 個の $\mathrm{SL}_2(\mathbb{Z})$ 不変量

$$\lambda^{ij}_{k\ell}(A,B) = \begin{vmatrix} a_{ij} & b_{ij} \\ a_{k\ell} & b_{k\ell} \end{vmatrix}$$

$(1,2,3)$ の置換 (i,j,k) に対して

$$\begin{aligned}
c^i_{ii} &= \pm\lambda^{ik}_{ij} + C_i, & c^j_{ii} &= \pm\lambda^{ii}_{ik},\\
c^i_{ij} &= \pm(1/2)\lambda^{ik}_{jj} + (1/2)C_j, & c^k_{ij} &= \pm\lambda^{jj}_{ii}.
\end{aligned}$$

$\pm$ は置換 (i,j,k) の符号，$C_1 = \lambda^{23}_{11}$, $C_2 = -\lambda^{13}_{22}$, $C_3 = \lambda^{12}_{33}$. $\Longrightarrow c^k_{ij}\in\mathbb{Z}$ $(k>0)$.
さらに

$$c^0_{ij} = \sum\nolimits_{r=1}^{3}(c^r_{jk}c^k_{ri} - c^r_{ij}c^k_{rk}).$$

4 次環 $Q(A,B)$ $\{\alpha_0 = 1, \alpha_1, \alpha_2, \alpha_3\}$ を基底とする自由 $\mathbb{Z}$ 加群に乗法を次のように定義したもの (Bhargava).

$$\alpha_i\alpha_j = \sum\nolimits_{k=0}^{3} c^k_{ij}\alpha_k \quad (i,j\in\{1,2,3\}).$$

$\mathrm{Disc}(Q(A,B)) = \mathrm{Disc}(F_{(A,B)})$.

§ 4

k : 非ガロア 3 次体，$\mathcal{O}$: k の整環，$f = (\mathcal{O}_k : \mathcal{O})$ 平方因数をもたないとする．

Disc$(k) > 0$ のとき

$$\begin{aligned} \sum_{x \in \Gamma \backslash L_1(\mathcal{O})} \mu(x) &= \sum_{g|f} |\operatorname{Cl}_{R_g}^{(2)}|(2 - |X(\mathcal{O}, R_g)|), \\ \sum_{x \in \Gamma \backslash L_1(\mathcal{O})} \mu(x) + \sum_{x \in \Gamma \backslash L_3(\mathcal{O})} \mu(x) &= \sum_{g|f} |\operatorname{Cl}_{R_g,+}^{(2)}|(2 - |X_+(\mathcal{O}, R_g)|), \end{aligned} \tag{4.7}$$

Disc$(k) < 0$ のとき

$$\sum_{x \in \Gamma \backslash L_2(\mathcal{O})} \mu(x) = \sum_{g|f} |\operatorname{Cl}_{R_g}^{(2)}|(2 - |X(\mathcal{O}, R_g)|). \tag{4.8}$$

命題 2.6, (4.7), (4.8) から非ガロア 3 次体 k の場合に定理 1 が得られる．ガロア 3 次体 k の場合も同様 (Aut$(\mathcal{O})$ が必ずしも自明でないから点に気をつける).

§ 5

命題 5.2. k_1 を 2 次体，$k = \mathbb{Q} \oplus k_1$, $\mathcal{O}_k = \mathbb{Z} \oplus \mathcal{O}_{k_1}$ とすれば

$$\begin{aligned} \sum_{x \in \Gamma \backslash L_1(\mathcal{O}_k)} \mu(x) &= \frac{1}{4} |\operatorname{Cl}_{k_1}^{(2)}| \quad (\mathrm{Disc}(k_1) > 0), \\ \sum_{x \in \Gamma \backslash L_2(\mathcal{O}_k)} \mu(x) &= \frac{1}{4} |\operatorname{Cl}_{k_1}^{(2)}| \quad (\mathrm{Disc}(k_1) < 0), \\ \sum_{x \in \Gamma \backslash L_1(\mathcal{O}_k)} \mu(x) + \sum_{x \in \Gamma \backslash L_3(\mathcal{O}_k)} \mu(x) &= \frac{1}{4} |\operatorname{Cl}_{k_1,+}^{(2)}| \quad (\mathrm{Disc}(k_1) > 0). \end{aligned}$$

一方，命題 2.6 より

$$\begin{aligned} \sum_{y \in \Gamma \backslash \hat{L}_1(\mathcal{O}_k)} \mu(y) &= \frac{1}{4} |\operatorname{Cl}_{k_1,+}^{(2)}| \quad (\mathrm{Disc}(k_1) > 0), \\ \sum_{y \in \Gamma \backslash \hat{L}_2(\mathcal{O}_k)} \mu(y) &= \frac{1}{2} |\operatorname{Cl}_{k_1}^{(2)}| \quad (\mathrm{Disc}(k_1) < 0), \\ \sum_{y \in \Gamma \backslash \hat{L}_1(\mathcal{O}_k)} \mu(y) + \sum_{y \in \Gamma \backslash \hat{L}_3(\mathcal{O}_k)} \mu(y) &= |\operatorname{Cl}_{k_1}^{(2)}| \quad (\mathrm{Disc}(k_1) > 0). \end{aligned}$$

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