Ohno Conjecture on the Zeta Functions associated with the Space of Binary Cubic Forms

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1 Introduction

Let $\Gamma = SL_2(\mathbb{Z})$ and let

$$x(u, v) = x_1 u^3 + x_2 u^2 v + x_3 uv^2 + x_4 v^3$$

be a binary cubic form with int. coeff..

The action of a matrix

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

is defined by

$$(\gamma x)(u, v) = x(au + cv, bu + dv).$$

The *discriminant* of x is defined by

$$\begin{aligned} D(x) &= 18x_1x_2x_3x_4 + x_2^2x_3^2 \\ &\quad -4x_1x_3^3 - 4x_2^3x_4 - 27x_1^2x_4^2. \end{aligned}$$

Then

$$D(\gamma x) = D(x), \quad \forall \gamma \in \Gamma.$$

Let

$$\begin{aligned} L &= \{x(u, v);\ x_i \in \mathbb{Z}\}, \\ \hat{L} &= \{x \in L; x_2, x_3 \in 3\mathbb{Z}\}. \end{aligned}$$

These sets are Γ-inv..

For any $n \in \mathbb{Z}$, $n \neq 0$, let

$$\begin{aligned} L(n) &= \{x \in L; D(x) = n\}, \\ \hat{L}(n) &= \{x \in \hat{L}; D(x) = n\}. \end{aligned}$$

We define the *class numbers*

$$\begin{aligned} h(n) &= \#\left(\Gamma\backslash L(n)\right), \\ \hat{h}(n) &= \#\left(\Gamma\backslash \hat{L}(n)\right). \end{aligned}$$

<u>Eisenstein, Arndt, Hermite, 19C</u>

$$h(n) < \infty, \quad \text{Tables}$$

To be more precise, let

$$\Gamma_x = \{\gamma \in \Gamma; \gamma x = x\}.$$

Then

$$|\Gamma_x| = \begin{cases} 1 \text{ or } 3, & D(x) > 0, \\ 1, & D(x) < 0. \end{cases}$$

According to the order of the isotropy subgroup, we define

$$\begin{aligned} h_1(n) &= \#\left(\Gamma\backslash\{x \in L(n); |\Gamma_x| = 1\}\right), \\ h_2(n) &= \#\left(\Gamma\backslash\{x \in L(n); |\Gamma_x| = 3\}\right). \end{aligned}$$

We define $\hat{h}_1(n)$ and $\hat{h}_2(n)$ similarly.

<u>Shintani, 1972.</u>

$$\begin{aligned} \xi_1(L,s) &= \sum_{n=1}^{\infty} \frac{h_1(n) + 3^{-1}h_2(n)}{n^s}, \\ \xi_2(L,s) &= \sum_{n=1}^{\infty} \frac{h(-n)}{n^s}, \\ \xi_1(\hat{L},s) &= \sum_{n=1}^{\infty} \frac{\hat{h}_1(n) + 3^{-1}\hat{h}_2(n)}{n^s}, \\ \xi_2(\hat{L},s) &= \sum_{n=1}^{\infty} \frac{\hat{h}(-n)}{n^s}. \end{aligned}$$

These Dirichlet series are abs. conv. for $\Re s > 1$, cont. to mero. func. on $\mathbb{C}$, only poles at $s = 1, \frac{5}{6}$ (simple), satisfy the func. eq.

$$\begin{pmatrix} \xi_1(L, 1-s) \\ \xi_2(L, 1-s) \end{pmatrix} = \Gamma\Big(s - \frac{1}{6}\Big)\Gamma(s)^2\Gamma\Big(s + \frac{1}{6}\Big)$$
$$\times 2^{-1}3^{6s-2}\pi^{-4s} \begin{pmatrix} \sin 2\pi s & \sin \pi s \\ 3 \sin \pi s & \sin 2\pi s \end{pmatrix} \begin{pmatrix} \xi_1(\hat{L}, s) \\ \xi_2(\hat{L}, s) \end{pmatrix}$$

Ohno Conjecture, 1995.

$$\begin{aligned} \text{(i)} \quad \xi_1(\hat{L}, s) &= 3^{-3s}\xi_2(L, s), \\ \text{(ii)} \quad \xi_2(\hat{L}, s) &= 3^{1-3s}\xi_1(L, s). \end{aligned}$$

We can rewrite the conjecture into the following relations of class numbers.

$$\begin{aligned} &\text{(i)}' \qquad \hat{h}_1(27n) + \frac{1}{3}\hat{h}_2(27n) = h(-n) \quad \forall n > 0; \\ &\text{(ii)}' \qquad \hat{h}(-27n) = 3h_1(n) + h_2(n) \quad \forall n > 0. \end{aligned}$$

The func. eq. implies (i) $\Longleftrightarrow$ (ii).

He also showed that under the conjecture, Diagonalization of func. eq. by Datskovsky–Wright implies simpler and more symmetric func. equ of a single zeta function:

$$Z_\pm(1-s) = Z_\pm(s),$$

where

$$\begin{aligned} Z_\pm(s) \quad = \quad & 2^s 3^{\frac{3}{2}s}\pi^{-2s} \\ & \times\Gamma(s)\Gamma\Big(\frac{s}{2} + \frac{1}{4} \mp \frac{1}{6}\Big)\Gamma\Big(\frac{s}{2} + \frac{1}{4} \mp \frac{1}{3}\Big) \\ & \times\Big(3^{\frac{1}{2}}\xi_1(L, s) \pm \xi_2(L, s)\Big). \end{aligned}$$

For simplicity, denote by $\tilde{h}(27n)$ the left hand side of (i)$'$:

$$\tilde{h}(27n) = \hat{h}_1(27n) + \frac{1}{3}\hat{h}_2(27n).$$

To prove the conjecture, it is enough to show

$$\tilde{h}(27n) = h(-n) \quad \forall n > 0.$$

By proving this equation directly, I succeeded to prove the conjecture.

Theorem 1. *The conjecture is true.*

2 Outline of the proof

Let $x \in \hat{L}(27n)$. We write

$$x(u,v) = x_1u^3 + 3x_2u^2v + 3x_3uv^2 + x_4v^3, \quad x_i \in \mathbb{Z}.$$

Let H_x be the Hessian of x.

$$H_x(u,v) = -\frac{1}{36}\begin{vmatrix} \dfrac{\partial^2 x}{\partial u^2} & \dfrac{\partial^2 x}{\partial u \partial v} \\ \dfrac{\partial^2 x}{\partial u \partial v} & \dfrac{\partial^2 x}{\partial v^2} \end{vmatrix}.$$

Then H_x is a positive definite integral binary quadratic form with disc. $-n$, and

$$H_{\gamma x} = \gamma H_x \quad (\forall \gamma \in \Gamma).$$

Let $k = \mathbb{Q}(\sqrt{-n})$. We now assume that $-n$ is a fund. disc., i.e. the disc. of k.

$$\begin{array}{ccc}\Gamma\backslash\{\text{bin. quad. forms with disc } -n\} & \longleftrightarrow & Cl_k \\ \bigcup & & \bigcup \\ \Gamma\backslash\{H_x; x\in\hat{L}(27n)\} & \longleftrightarrow & Cl_k^{(3)}\end{array}$$

$$Cl_k^{(3)} = \{c \in Cl_k; c^3 = 1\}.$$

Hence

$$\tilde{h}(27n) = |Cl_k^{(3)}|.$$

Datskovsky–Wright, 1986

$$\begin{aligned}\frac{1}{2}\xi_2(L,s) &= \sum_{K:\text{cubic f.},D_K<0} |D_K|^{-s}\eta_K(2s) \\ &\quad + \frac{1}{2}\sum_{k:\text{imag. quad. f.}} |D_k|^{-s}\eta_{\mathbb{Q}\oplus k}(2s),\end{aligned}$$

where

$$\eta_A(s) = \zeta(2s)\zeta(3s-1)\frac{\zeta_A(s)}{\zeta_A(2s)},$$

$$\zeta_A(s) = \prod_i \zeta_{K_i}(s), \quad A = \oplus_i K_i.$$

This expression implies that

$$\begin{aligned}h(-n) &= 2\#\{\text{cubic fields with disc.} -n\} + 1 \\ &= |Cl_k^{(3)}|\end{aligned}$$

Thus we have

$$\tilde{h}(27n) = h(-n)$$

under the assumption that $-n$ is a fund. disc..

The case $-n = m^2 D_k$, m:square free, is proved by generalizing the argument above. The case of arbitrary m is proved by some recursive formulae for $h(-np^{2r})$ and $\hat{h}(27np^{2r})$, $r = 0, 1, 2, \ldots$ coming from D-W's expression.

3 Application

Let $N_3(n)$ be the number of the cubic fields with discriminant n.

Theorem 2. *Let k be an imaginary quadratic field with $k \neq \mathbb{Q}(\sqrt{-3})$ and put $n = |D_k|$. If $3 \nmid n$, then*

$$\begin{aligned} N_3(3n) + N_3(27n) &= N_3(-n), \\ N_3(-n) + N_3(-81n) &= 3N_3(3n) + 1. \end{aligned}$$

If $3|n$, then

$$\begin{aligned} N_3(n/3) + N_3(27n) &= N_3(-n), \\ N_3(-n) + N_3(-9n) &= 3N_3(n/3) + 1. \end{aligned}$$

For any quadratic field k and for any positive integer c, denote by $\mathcal{O}_{k,c}$ the order of k of conductor c, and denote by $r_{k,c}$ the 3-rank of the ideal class group of $\mathcal{O}_{k,c}$. By class field theory, Theorem 2 is equivalent to the following

Theorem 3. *Let k and n be as in Theorem 2 and let k' be the real quadratic field $\mathbb{Q}(\sqrt{3n})$. If $3 \nmid n$, then $r_{k',3} = r_{k,1}$ and $r_{k,9} = r_{k',1} + 1$. If $3|n$, then $r_{k',9} = r_{k,1}$ and $r_{k,3} = r_{k',1} + 1$.*

Remark 4. *Theorem 3 can be viewed as a precise version of Scholz's reflection theorem.*

Remark 5. *The residue of $\xi_2(L, s)$ at $s = \frac{5}{6}$ is equal to that of $\sqrt{3}\xi_1(L, s)$. Hence $Z_-(s)$ has only one pole at at $s = 1$, while $Z_+(s)$ has exactly two poles at $s = 1$ and $s = \frac{5}{6}$.*

Remark 6. *If the direct bijection between classes in question can be easily described in some way, it should be very interesting. However, I have no idea on this. In general, the number of the equivalence classes of irreducible forms in $\hat{L}(27n)$ does not coincide with that of irreducible forms in $L(-n)$.*