A conjecture on the zeta functions of pairs of ternary quadratic forms

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Pairs of ternary quadratic forms

V: the vector space of pairs of ternary quadratic forms $/\mathbb{C}$ in $v = (v_1, v_2, v_3)$.

$$x = (x_1, x_2) \in V, \quad x_k(v) = \sum_{1 \le i \le j \le 3} x_{k,ij} v_i v_j \qquad (k = 1, 2)$$

Identify x_k by symmetric matrix. Define a binary cubic form $F_x(u)$ in $u = (u_1, u_2)$ by

$$F_x(u) = 4 \det(u_1 x_1 - u_2 x_2) = a u_1^3 + b u_1^2 u_2 + c u_1 u_2^2 + d u_2^3.$$

a, b, c, d are homogeneous polynomials of $x_{k,ij}$ of degree 3. Put

$$\mathrm{Disc}(x) = \mathrm{Disc}(F_x) = 18abcd + b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2.$$

$\mathrm{Disc}(x)$ is a homogeneous polynomial of $x_{k,ij}$ of degree 12.

Group action

The group $G = \mathrm{SL}_3(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$ acts on V:
$g = (g_1, g_2) \in G$, $x = (x_1, x_2) \in V$,

$$g \cdot x = (p(g_1 x_1) + q(g_1 x_2), r(g_1 x_1) + s(g_1 x_2)),$$

$g_2 = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, $(g_1 x_k)(v) = x_k(v g_1)$. Then we have

$$\mathrm{Disc}(g \cdot x) = (\det g_1)^8 (\det g_2)^6 \,\mathrm{Disc}(x).$$

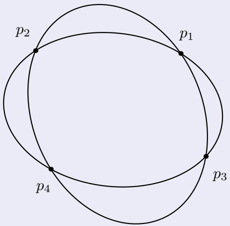
$S = \{x \in V \ : \ \mathrm{Disc}(x) = 0\}$ is an irreducible hypersurface,
$V \setminus S$ is a single G-orbit.
(G, V) is a prehomogeneous vector space,
$\mathrm{Disc}(x)$ is its fundamental relative invariant.

Zero(x)

For any $x \in V \setminus S$, put

$$\mathrm{Zero}(x) = \{v \in \mathbb{P}^2(\mathbb{C}) \ : \ x_1(v) = x_2(v) = 0\}.$$

Then $\mathrm{Zero}(x)$ is a finite set consisting of four points.



The set $V_{\mathbb{R}} \setminus S_{\mathbb{R}}$ decomposes into three $G_{\mathbb{R}}$-orbits V_1, V_2, V_3. V_1, V_2 and V_3 are the set of $x \in V_{\mathbb{R}} \setminus S_{\mathbb{R}}$ such that the cardinality of $\mathrm{Zero}(x) \cap \mathbb{P}^2(\mathbb{R})$ equals 4, 2, and 0, respectively.

Dual lattice

$L \subset V_{\mathbb{R}}$: the lattice of pairs of integral ternary quadratic forms.
For $x = (x_1, x_2), y = (y_1, y_2) \in V$,

$$\langle x, y \rangle = \mathrm{Tr}(x_1 y_2) + \mathrm{Tr}(x_2 y_1) \qquad (\mathrm{Tr} \text{ is the trace of matrices}).$$

$\hat{L}$: the dual lattice of L w.r.t. $\langle\ ,\ \rangle$
is the set of pairs of integral symmetric matrices of degree 3.
$\Gamma_1 = \mathrm{SL}_3(\mathbb{Z})$, $\Gamma_2 = \mathrm{GL}_2(\mathbb{Z})$. $\Gamma = \Gamma_1 \times \Gamma_2$ acts on L and $\hat{L}$.
For $y = (y_1, y_2) \in \hat{L}$, define an integral binary cubic form

$$\hat{F}_y(u) = \det(u_1 y_1 - u_2 y_2) = (1/4) F_y(u).$$

Put

$$\mathrm{Disc}^*(y) = \mathrm{Disc}(\hat{F}_y) = 2^{-8}\,\mathrm{Disc}(F_y).$$

Zeta functions

Γ_x: the isotropy group of $x \in L \setminus S$ in Γ. $\quad |\Gamma_x| \leq 72$.
Put $\mu(x) = 1/|\Gamma_x|$.
The zeta functions $\xi_i(L, s)$, $\xi_i(\hat{L}, s)$ $(i = 1, 2, 3)$ are define by

$$\xi_i(L, s) = \sum_{x \in \Gamma \backslash L \cap V_i} \frac{\mu(x)}{|\operatorname{Disc}(x)|^s} = \sum_{n=1}^{\infty} \frac{a_i((-1)^{i-1}n)}{n^s},$$

$$\xi_i(\hat{L}, s) = \sum_{y \in \Gamma \backslash \hat{L} \cap V_i} \frac{\mu(y)}{|\operatorname{Disc}^*(y)|^s} = \sum_{n=1}^{\infty} \frac{\hat{a}_i((-1)^{i-1}n)}{n^s},$$

where

$$a_i(n) = \sum_{\substack{x \in \Gamma \backslash (L \cap V_i) \\ \operatorname{Disc}(x) = n}} \mu(x), \qquad \hat{a}_i(n) = \sum_{\substack{y \in \Gamma \backslash (\hat{L} \cap V_i) \\ \operatorname{Disc}^*(y) = n}} \mu(y).$$

Functional equations

The zeta functions $\xi_i(L, s)$, $\xi_i(\hat{L}, s)$ converge absolutely for $\Re(s) > 1$. (Yukie, '*Shintani Zeta functions*', Cambridge Univ. Press, 1993). The functional equations

$$\begin{aligned}
&(\xi_i(L, 1-s)) \\
&= \Gamma(s)^4 \Gamma\left(s - \frac{1}{6}\right)^2 \Gamma\left(s + \frac{1}{6}\right)^2 \Gamma\left(s - \frac{1}{4}\right)^2 \Gamma\left(s + \frac{1}{4}\right)^2 \\
&\times 2^{8s} 3^{6s} \pi^{-12s} (u_{ji}^*(s))(\xi_j(\hat{L}, s)).
\end{aligned}$$

hold, where $u_{ji}^*(s)$' are polynomials of $q = \exp(\pi\sqrt{-1}s)$ and q^{-1} of degree at most 6. (Sato-Shintani, On zeta functions associated with prehomogeneous vector spaces, *Ann. of Math.* **100** (1974), 131–170).

Conjecture

We present the following conjecture which is a quartic analogue of Ohno conjecture (Y. Ohno, A conjecture on coincidence among the zeta functions associated with the space of binary cubic forms, *Amer. J. Math*. **119** (1997), 1083–1094.
J. Nakagawa, On the relations among the class numbers of binary cubic forms, Invent. math. 134, 101-138 (1998)).

Conjecture 1.1

$$\begin{aligned}
\xi_1(\hat{L}, s) &= \xi_1(L, s) + \xi_3(L, s), \\
\xi_2(\hat{L}, s) &= 2\xi_2(L, s), \\
\xi_3(\hat{L}, s) &= 3\xi_1(L, s) - \xi_3(L, s).
\end{aligned}$$

Results

a ring of rank n: a comm. ring with 1 that is a free $\mathbb{Z}$-module of rank n.

It is called *nondegenerate* if its discriminant is non-zero.

For any $F(u) = au_1^3 + bu_1^2u_2 + cu_1u_2^2 + du_2^3 \in \mathbb{Z}[u_1, u_2]$,

$R(F)$: the cubic ring associated with $F(u)$ (Delone-Faddeev).

$R(F)$ is a free $\mathbb{Z}$-module having $\mathbb{Z}$-basis $\{1, \omega, \theta\}$ and the multiplication table

$$\begin{aligned} \omega^2 &= -ac + b\omega - a\theta, \\ \theta^2 &= -bd + d\omega - c\theta, \\ \omega\theta &= -ad. \end{aligned} \tag{1.1}$$

$F \mapsto R(F)$ gives a discriminant preserving bijection

$$\mathrm{GL}_2(\mathbb{Z})\backslash\{\text{integral binary cubic forms}\} \longleftrightarrow \{\text{cubic rings}\}/\cong .$$

For any nondeg. cubic ring $\mathcal{O}$ and $i = 1, 2, 3$, put

$$L(\mathcal{O}) = \{x \in L : R(F_x) \cong \mathcal{O}\}, \quad L_i(\mathcal{O}) = L(\mathcal{O}) \cap V_i$$
$$\hat{L}(\mathcal{O}) = \{y \in \hat{L} : R(\hat{F}_y) \cong \mathcal{O}\}, \quad \hat{L}_i(\mathcal{O}) = \hat{L}(\mathcal{O}) \cap V_i.$$

$\mathcal{O}_k$: the maximal order of an étale algebra k over $\mathbb{Q}$.

Theorem 1

Let k be a cubic field and $\mathcal{O}$ be an order of k such that the index $(\mathcal{O}_k : \mathcal{O})$ is square free. Then the following relations hold:

$$\sum_{y \in \Gamma \backslash \hat{L}_1(\mathcal{O})} \mu(y) = \sum_{x \in \Gamma \backslash L_1(\mathcal{O})} \mu(x) + \sum_{x \in \Gamma \backslash L_3(\mathcal{O})} \mu(x) \quad (\mathrm{Disc}(k) > 0),$$

$$\sum_{y \in \Gamma \backslash \hat{L}_2(\mathcal{O})} \mu(y) = 2 \sum_{x \in \Gamma \backslash L_2(\mathcal{O})} \mu(x) \quad (\mathrm{Disc}(k) < 0),$$

$$\sum_{y \in \Gamma \backslash \hat{L}_3(\mathcal{O})} \mu(y) = 3 \sum_{x \in \Gamma \backslash L_1(\mathcal{O})} \mu(x) - \sum_{x \in \Gamma \backslash L_3(\mathcal{O})} \mu(x) \quad (\mathrm{Disc}(k) > 0).$$

By Theorem 1 and applying Gauss's genus theory on the 2-rank of the ideal class groups of quadratic fields for the reducible algebras $k = \mathbb{Q} \oplus \mathbb{Q}(\sqrt{n})$, we obtain

Theorem 2

If n is a discriminant of a quadratic field, then the following relations hold:

$$\begin{aligned} \hat{a}_1(n) &= a_1(n) + a_3(n) \quad (n > 0), \\ \hat{a}_2(n) &= 2a_2(n) \quad (n < 0), \\ \hat{a}_3(n) &= 3a_1(n) - a_3(n) \quad (n > 0). \end{aligned}$$

HCL II

$(\mathcal{O}, \mathfrak{a}, \delta)$: $\mathcal{O}$ is a nondeg. cubic ring, $\mathfrak{a}$ is a frac. ideal of $\mathcal{O}$,

$$\delta \in k^\times,\ k = \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Q},\ \mathfrak{a}^2 \subset \delta\mathcal{O},\ N_{k/\mathbb{Q}}(\delta) = N_{\mathcal{O}}(\mathfrak{a})^2.$$

$N_{\mathcal{O}}(\mathfrak{a})$: the norm of $\mathfrak{a}$ as a frac. $\mathcal{O}$-ideal,

$$N_{\mathcal{O}}(\mathfrak{a}) = (\mathcal{O} : \mathfrak{a}) \text{ for } \mathfrak{a} \subset \mathcal{O}.$$

$$(\mathcal{O}, \mathfrak{a}, \delta) \sim (\mathcal{O}', \mathfrak{a}', \delta')$$
$$\iff \exists \phi : \mathcal{O} \cong \mathcal{O}',\ \exists \kappa \in \mathcal{O}' \otimes_{\mathbb{Z}} \mathbb{Q},\ \mathfrak{a}' = \kappa\phi(\mathfrak{a}),\ \delta' = \kappa^2\phi(\delta).$$

M. Bhargava proved the following theorem (Higher composition laws II, *Ann. of Math*. **159** (2004), 865–886).

Thorem 2.1 (Bhargava)

$$\Gamma\backslash\{y \in \hat{L} : \mathrm{Disc}^*(y) \neq 0\} \longleftrightarrow \{(\mathcal{O}, \mathfrak{a}, \delta)\}/\sim,$$
$$\Gamma y \longmapsto [\mathcal{O}, \mathfrak{a}, \delta] \Longrightarrow \mathrm{Disc}(\mathcal{O}) = \mathrm{Disc}^*(y).$$

Bijection of Theorem 2.1

$\mathcal{O}$: a nondeg. cubic ring, with $\mathbb{Z}$-basis $\{1, \omega, \theta\}$
multiplication table (1.1), $a, b, c, d \in \mathbb{Z}$, $k = \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Q}$,
$\mathfrak{a}$: a frac. ideal of $\mathcal{O}$, $\delta \in k^\times$, $\mathfrak{a}^2 \subset \delta\mathcal{O}$, $N_{k/\mathbb{Q}}(\delta) = N_{\mathcal{O}}(\mathfrak{a})^2$,
$\{\alpha_1, \alpha_2, \alpha_3\}$: a $\mathbb{Z}$-basis of $\mathfrak{a}$,

$$(\alpha_1, \alpha_2, \alpha_3) = (1, \omega, \theta)\gamma_1 \ (\exists \gamma_1 \in \mathrm{GL}_3(\mathbb{Q}),\ \det \gamma_1 > 0).$$

$\mathfrak{a}^2 \subset \delta\mathcal{O} \Longrightarrow \exists a_{ij}, b_{ij}, c_{ij} \in \mathbb{Z}$ such that

$$\alpha_i \alpha_j = \delta(c_{ij} + b_{ij}\omega + a_{ij}\theta). \tag{2.1}$$

Put $A = (a_{ij})$, $B = (b_{ij})$. Then

$$\hat{F}_{(A,B)}(u) = au_1^3 + bu_1^2 u_2 + cu_1 u_2^2 + du_2^3.$$

$(\mathcal{O}, \mathfrak{a}, \delta) \mapsto \Gamma \cdot (A, B)$ gives the bijection of Theorem 2.1.

Isotropy group $\Gamma_{(A,B)}$

$\Gamma_{(A,B)}$: the isotropy group in Γ of a nondegen. pair $(A, B) \in \hat{L}$.

Corollary 2.2 (Bhargava)

For any nondeg. pair $(A, B) \in \hat{L}$,

$$\exists \Gamma_{(A,B)} \to \mathrm{Aut}(\mathcal{O}) \quad (\textit{group homo.})$$
$$\textit{with kernel } \cong U_2^+(\mathcal{O}_0).$$

Here $(\mathcal{O}, \mathfrak{a})$ *is the pair corresponding to* (A, B) *as in Theorem 2.1,* $\mathcal{O}_0 = \mathrm{End}_{\mathcal{O}}(\mathfrak{a})$, $U_2^+(\mathcal{O}_0) = \{\varepsilon \in \mathcal{O}_0^\times : \varepsilon^2 = 1,\ N_{k/\mathbb{Q}}(\varepsilon) > 0\}$.

$\hat{L}(\mathcal{O}, \mathcal{O}_0)$

k: an étale cubic algebra over $\mathbb{Q}$,
$\mathcal{O}$: an order of k such that $f = (\mathcal{O}_k : \mathcal{O})$ is square free.
$\hat{L}(\mathcal{O}) = \{(A, B) \in \hat{L} \ : \ R(\hat{F}_{(A,B)}) \cong \mathcal{O}\}$.
Then

$$\hat{L}(\mathcal{O}) = \bigcup_{\mathcal{O} \subset \mathcal{O}_0 \subset \mathcal{O}_k} \hat{L}(\mathcal{O}, \mathcal{O}_0),$$

$$\hat{L}(\mathcal{O}, \mathcal{O}_0) = \{(A, B) \in \hat{L}(\mathcal{O}) \ : \ \mathrm{End}_{\mathcal{O}}(\mathfrak{a}) = \mathcal{O}_0\},$$

$$\Gamma \cdot (A, B) \longleftrightarrow [\mathcal{O}, \mathfrak{a}, \delta].$$

Γ_1-orbits and Γ-orbits

Put

$$\hat{L}(F, \mathcal{O}_0) = \{(A, B) \in \hat{L}(\mathcal{O}, \mathcal{O}_0) \ : \ \hat{F}_{(A,B)} = F\}.$$

Then $\Gamma_1 \cdot (A, B) \longmapsto \Gamma \cdot (A, B)$ defines a surjective mapping

$$\Gamma_1 \backslash \hat{L}(F, \mathcal{O}_0) \longrightarrow \Gamma \backslash \hat{L}(\mathcal{O}, \mathcal{O}_0).$$

By Corollary 2.2, the cardinality of the inverse image of $\Gamma \cdot (A, B)$ is $|\operatorname{Aut}(\mathcal{O})| \cdot |U_2^+(\mathcal{O}_0)|/|\Gamma_{(A,B)}|$. Hence

Lemma 2.3

$$|\Gamma_1 \backslash \hat{L}(F, \mathcal{O}_0)| = \sum_{(A,B) \in \Gamma \backslash \hat{L}(\mathcal{O}, \mathcal{O}_0)} \frac{|\operatorname{Aut}(\mathcal{O})| \cdot |U_2^+(\mathcal{O}_0)|}{|\Gamma_{(A,B)}|}.$$

Ideal $\mathfrak{j}(\mathcal{O}, \mathcal{O}_0)$

$\exists\{1, \omega, \theta\}$: a basis of $\mathcal{O}_k$, $\omega, \theta \in k^\times$, $\mathcal{O} = [1, f\omega, \theta]$,
multiplication table (1.1), $a, b, c, d \in \mathbb{Z}$ $\quad (\Rightarrow f|d)$.
$(\mathcal{O}, \mathfrak{a}, \delta)$: a triplet, $\mathcal{O}_0 = \mathrm{End}_{\mathcal{O}}(\mathfrak{a})$.
$\Longrightarrow \mathcal{O}_0 = [1, g\omega, \theta]$ $(\exists g|f)$. Write $f = gh$.
$\mathfrak{f} = [f, f\omega, \theta]$: the conductor of $\mathcal{O}$ (the maximal $\mathcal{O}_k$-ideal in $\mathcal{O}$)
$\mathfrak{g} = [g, g\omega, \theta]$: the conductor of $\mathcal{O}_0$.
$\Longrightarrow \mathcal{O} = \mathbb{Z} + \mathfrak{f}$, $\mathcal{O}_0 = \mathbb{Z} + \mathfrak{g}$.
Put $\mathfrak{j} = [h, f\omega, \theta] \subset \mathcal{O}$. Then $\mathfrak{j}$ is an $\mathcal{O}_0$-ideal.
$f = gh$ is square free $\Longrightarrow \gcd(g, h) = 1$, $\mathfrak{j} + \mathfrak{g} = \mathcal{O}_0$.
$\mathfrak{h} = [h, h\omega, \theta]$ and $\mathfrak{h}' = [h, \omega, \theta + c]$ are $\mathcal{O}_k$-ideals, $\mathfrak{h}\mathfrak{h}' = h\mathcal{O}_k$.
$\mathfrak{j}\mathcal{O}_k = \mathfrak{h} \Longrightarrow \mathfrak{j}$ is an invertible $\mathcal{O}_0$-ideal.
$\mathfrak{j}$ is the largest $\mathcal{O}_0$-ideal in $\mathcal{O}$. Write $\mathfrak{j} = \mathfrak{j}(\mathcal{O}, \mathcal{O}_0)$.

Fröhlich's result

R: a nondeg. ring of rank n. $\mathfrak{c}$: a frac. ideal of R.
$\mathfrak{c}$ is *proper* $\iff \mathrm{End}_R(\mathfrak{c}) = R$.
R is *weakly self dual* $\iff$ every proper R-ideal is invertible.
$g = (\mathcal{O}_k : \mathcal{O}_0)$ is square free $\Longrightarrow \mathcal{O}_0$ is weakly self dual.
(Fröhlich, Invariants for modules over commutative separable orders, Quart. J. Math. Oxford (2) **16** (1965), 193–232.).
Hence $\mathfrak{a}$ is an invertible $\mathcal{O}_0$-ideal.
By a result of Fröhlich, for any proper $\mathcal{O}_0$-ideal $\mathfrak{b}$

$$\mathfrak{b} \text{ is invertible } \iff (\mathcal{O}_k\mathfrak{b} : \mathfrak{b}) = (\mathcal{O}_k : \mathcal{O}_0).$$

Hence for any proper integral $\mathcal{O}_0$-ideal $\mathfrak{b}$, we have

$$N_{\mathcal{O}_0}(\mathfrak{b}) = (\mathcal{O}_0 : \mathfrak{b}) = \frac{(\mathcal{O}_k : \mathcal{O}_k\mathfrak{b})(\mathcal{O}_k\mathfrak{b} : \mathfrak{b})}{(\mathcal{O}_k : \mathcal{O}_0)} = (\mathcal{O}_k : \mathcal{O}_k\mathfrak{b}) = N(\mathcal{O}_k\mathfrak{b}).$$

$\delta^{-1}\mathfrak{a}^2 = \mathfrak{j}(\mathcal{O}, \mathcal{O}_0)$

This implies that the norm $N_{\mathcal{O}_0}$ is multiplicative for proper $\mathcal{O}_0$-ideals. Hence

$$N_{\mathcal{O}_0}(\mathfrak{a}) = (\mathcal{O}_0 : \mathcal{O}) N_{\mathcal{O}}(\mathfrak{a}) = h N_{\mathcal{O}}(\mathfrak{a}),$$
$$N_{\mathcal{O}_0}(\delta^{-1}\mathfrak{a}^2) = N_{k/\mathbb{Q}}(\delta)^{-1} N_{\mathcal{O}_0}(\mathfrak{a})^2 = N_{k/\mathbb{Q}}(\delta)^{-1} h^2 N_{\mathcal{O}}(\mathfrak{a})^2 = h^2.$$

$\mathfrak{j} = [h, gh\omega, \theta]$, $\mathcal{O}_0 = [1, g\omega, \theta] \Longrightarrow N_{\mathcal{O}_0}(\mathfrak{j}) = h^2 = N_{\mathcal{O}_0}(\delta^{-1}\mathfrak{a}^2)$.
$\delta^{-1}\mathfrak{a}^2 \subset \mathfrak{j} \Longrightarrow \delta^{-1}\mathfrak{a}^2 = \mathfrak{j}$.
$I_{\mathcal{O}_0}$: the group of invertible frac. $\mathcal{O}_0$-ideals.
Put

$$\mathscr{I}(\mathcal{O}, \mathcal{O}_0) = \{(\mathfrak{a}, \delta) \in I_{\mathcal{O}_0} \times k^\times : \delta^{-1}\mathfrak{a}^2 = \mathfrak{j}(\mathcal{O}, \mathcal{O}_0), N_{k/\mathbb{Q}}(\delta) > 0\}.$$

For any $(\mathfrak{a}, \delta)$, $(\mathfrak{a}', \delta') \in \mathscr{I}(\mathcal{O}, \mathcal{O}_0)$,

$$(\mathfrak{a}, \delta) \sim (\mathfrak{a}', \delta') \Longleftrightarrow \exists \kappa \in k^\times, \mathfrak{a}' = \kappa\mathfrak{a}, \delta' = \kappa^2\delta.$$

Then $(\mathfrak{a}, \delta) \longmapsto \Gamma_1 \cdot (A, B)$ induces a bijection

$$\mathscr{I}(\mathcal{O}, \mathcal{O}_0)/\sim \;\longleftrightarrow\; \Gamma_1\backslash\hat{L}(F, \mathcal{O}_0),$$

$$([\mathcal{O}, \mathfrak{a}, \delta] \longleftrightarrow \Gamma \cdot (A, B) \text{ in Theorem 2.1}).$$

$X(\mathcal{O}, \mathcal{O}_0) \subset \mathrm{Cl}_{\mathcal{O}_0} / \mathrm{Cl}_{\mathcal{O}_0}^2$: the subgr. gen. by $\mathfrak{j}(\mathcal{O}, \mathcal{O}_0)$.

$$\mathscr{I}(\mathcal{O}, \mathcal{O}_0) \neq \emptyset \Longleftrightarrow |X(\mathcal{O}, \mathcal{O}_0)| = 1.$$

$\mathrm{Cl}_{\mathcal{O}_0}^{(2)} = \{c \in \mathrm{Cl}_{\mathcal{O}_0} : c^2 = 1\}$. Then

$$|\mathscr{I}(\mathcal{O}, \mathcal{O}_0)/\sim| = |\,\mathrm{Cl}_{\mathcal{O}_0}^{(2)}\,|(U^+(\mathcal{O}_0) : U^+(\mathcal{O}_0)^2)\,(2 - |X(\mathcal{O}, \mathcal{O}_0)|).$$

By Lemma 2.3,

$$\sum_{(A,B)\in\Gamma\backslash\hat{L}(\mathcal{O},\mathcal{O}_0)} \frac{1}{|\Gamma_{(A,B)}|} = \frac{(U^+(\mathcal{O}_0) : U^+(\mathcal{O}_0)^2)}{|\,\mathrm{Aut}(\mathcal{O})|\,|U_2^+(\mathcal{O}_0)|}\,|\,\mathrm{Cl}_{\mathcal{O}_0}^{(2)}\,| \times (2 - |X(\mathcal{O}, \mathcal{O}_0)|).$$

Assume $\mathrm{Disc}(k) > 0$.

$$\hat{L}(\mathcal{O}, \mathcal{O}_0) = \hat{L}_1(\mathcal{O}, \mathcal{O}_0) \cup \hat{L}_3(\mathcal{O}, \mathcal{O}_0).$$

$\mathrm{Cl}_{\mathcal{O}_0,+}$: the narrow ideal class group.
$U_+(\mathcal{O}_0)$: the group of totally positive units in $\mathcal{O}_0$,
$U_2(\mathcal{O}_0) = \{\varepsilon \in \mathcal{O}_0^\times : \varepsilon^2 = 1\}$,
$X_+(\mathcal{O}, \mathcal{O}_0) \subset \mathrm{Cl}_{\mathcal{O}_0,+} / \mathrm{Cl}^2_{\mathcal{O}_0,+}$: the subgr. gen. by $\mathfrak{j}(\mathcal{O}, \mathcal{O}_0)$.

$$\sum_{(A,B)\in\Gamma\backslash\hat{L}_1(\mathcal{O},\mathcal{O}_0)} \frac{1}{|\Gamma_{(A,B)}|} = \frac{(U_+(\mathcal{O}_0) : U_+(\mathcal{O}_0)^2)}{2^3|\mathrm{Aut}(\mathcal{O})|\,|U_2(\mathcal{O}_0)|}\,|\,\mathrm{Cl}^{(2)}_{\mathcal{O}_0,+}\,| \times (2 - |X_+(\mathcal{O}, \mathcal{O}_0)|).$$

Proposition 2.4

Let k be a cubic field and $\mathcal{O} \subset \mathcal{O}_0$ be orders of k such $(\mathcal{O}_k : \mathcal{O})$ is square free. If $\mathrm{Disc}(k) > 0$, then

$$\sum\nolimits_{y \in \Gamma \backslash \hat{L}_1(\mathcal{O},\mathcal{O}_0)} \mu(y) = \frac{|\,\mathrm{Cl}^{(2)}_{\mathcal{O}_0,+}\,|}{|\,\mathrm{Aut}(\mathcal{O})|}(2 - |X_+(\mathcal{O},\mathcal{O}_0)|),$$

$$\sum\nolimits_{y \in \Gamma \backslash \hat{L}_1(\mathcal{O},\mathcal{O}_0)} \mu(y) + \sum\nolimits_{y \in \Gamma \backslash \hat{L}_3(\mathcal{O},\mathcal{O}_0)} \mu(y) = \frac{4|\,\mathrm{Cl}^{(2)}_{\mathcal{O}_0}\,|}{|\,\mathrm{Aut}(\mathcal{O})|}(2 - |X(\mathcal{O},\mathcal{O}_0)|),$$

otherwise

$$\sum_{y \in \Gamma \backslash \hat{L}_2(\mathcal{O},\mathcal{O}_0)} \mu(y) = 2|\,\mathrm{Cl}^{(2)}_{\mathcal{O}_0}\,|\,(2 - |X(\mathcal{O},\mathcal{O}_0)|).$$

SL_2-invariants of pairs of ternary quadratic forms

Write $(A, B) \in L$ as

$$A(v) = \sum_{1 \le i \le j \le 3} a_{ij} v_i v_j, \quad B(v) = \sum_{1 \le i \le j \le 3} b_{ij} v_i v_j.$$

Put $a_{ji} = a_{ij}$, $b_{ji} = b_{ij}$ and define $\lambda^{ij}_{k\ell} = \lambda^{ij}_{k\ell}(A, B)$ by

(3.1) $$\lambda^{ij}_{k\ell}(A, B) = \begin{vmatrix} a_{ij} & b_{ij} \\ a_{k\ell} & b_{k\ell} \end{vmatrix}.$$

For any permutation (i, j, k) of $(1, 2, 3)$, we put

$$\begin{aligned} c^i_{ii} &= \pm\lambda^{ik}_{ij} + C_i, & c^j_{ii} &= \pm\lambda^{ii}_{ik}, \\ c^i_{ij} &= \pm(1/2)\lambda^{ik}_{jj} + (1/2)C_j, & c^k_{ij} &= \pm\lambda^{jj}_{ii}, \end{aligned}$$

$C_1 = \lambda^{23}_{11}$, $C_2 = -\lambda^{13}_{22}$, $C_3 = \lambda^{12}_{33}$, $\pm$ denotes the signature of the permutation (i, j, k). Then $c^k_{ij} \in \mathbb{Z}$ for $k > 0$. Put

(3.2) $$c^0_{ij} = \sum_{r=1}^{3} (c^r_{jk} c^k_{ri} - c^r_{ij} c^k_{rk}).$$

Quartic rings and pairs of ternary quadratic forms

$Q(A,B)$: the quartic ring associated with (A,B). (Bhargava)
$Q(A,B)$ is a free $\mathbb{Z}$-module with basis $\{\alpha_0 = 1, \alpha_1, \alpha_2, \alpha_3\}$, and the multiplication of $Q(A,B)$ is given by

$$(3.3) \qquad \alpha_i \alpha_j = \sum_{k=0}^{3} c_{ij}^k \alpha_k \quad (i, j \in \{1,2,3\}).$$

The discriminant of $Q(A,B)$ is equal to $\mathrm{Disc}(A,B) = \mathrm{Disc}(F_{(A,B)})$, $F_{(A,B)}(u) = 4\det(u_1 A - u_2 B)$. The cubic ring $R(A,B) = R(F_{(A,B)})$ has the same discriminant $\mathrm{Disc}(A,B)$.
For any quartic ring Q and for an element $x \in Q$, denote by x, x', x'', x''' the 'conjugates' of x. Put $\phi(x) = xx' + x''x'''$. Then all $\phi(x)$ are contained in some cubic ring $R^{\mathrm{inv}}(Q)$. A *cubic resolvent ring* of Q is a cubic ring R such that $\mathrm{Disc}(R) = \mathrm{Disc}(Q)$ and $R \supset R^{\mathrm{inv}}(Q)$.

HCL III

Bhargava proved the following theorem (Higher composition laws III, *Ann. of Math.* **159** (2004), 1329–1360.).

Thorem 3.1 (Bhargava)

$(A, B) \mapsto (Q(A, B), R(A, B))$ *induces a bijection*

$$\Gamma \backslash \{(A, B) \in L : \mathrm{Disc}(A, B) \neq 0\} \longleftrightarrow \{(Q, R)\}/\cong,$$

Q *is a nondeg. quartic ring,* R *is a cubic resolvent ring of* Q.

Corollary 3.2 (Bhargava)

Every quartic ring has a cubic resolvent ring. A primitive quartic ring has a unique cubic resolvent ring up to isomorphism. In particular, every maximal quartic ring has a unique cubic resolvent ring.

$\Gamma_{(A,B)}$ and $\mathrm{Aut}(Q(A,B))$

We call (A,B) *primitive* if $\gcd(\lambda^{ij}_{k\ell}(A,B)) = 1$.

Proposition 3.3

For any nondeg. pair $(A,B) \in L$,

$$\exists\, \Gamma_{(A,B)} \longrightarrow \mathrm{Aut}(Q(A,B)) \quad (\textit{inj. group homo.}).$$

If (A,B) *is primitive, then the homo. is an isom.*

Hence if (A,B) is primitive,

$$\mu(A,B) = \frac{1}{|\Gamma_{(A,B)}|} = \frac{1}{|\,\mathrm{Aut}(Q(A,B))|}.$$

Relative discriminant $\mathrm{Disc}(k_6/k)$

k: a non-Galois cubic field,
$\mathcal{O}$: an order of k, $f = (\mathcal{O}_k : \mathcal{O})$ is square free,
K: an S_4-quartic field,
K contains an order Q that has a cubic resolvent ring $\cong \mathcal{O}$,
$\tilde{K}$: the Galois closure of K,
k_6: the non-Galois sextic field, $k \subset k_6 \subset \tilde{K}$.
$k_6 = k(\sqrt{\alpha})$, $\alpha \in k^\times$, $N_{k/\mathbb{Q}}(\alpha) = a^2$, $a \in \mathbb{Q}^\times$.
$\Longrightarrow N(\mathrm{Disc}(k_6/k)) = g^2$ $(\exists g \in \mathbb{N})$, $\mathrm{Disc}(K) = g^2\,\mathrm{Disc}(k)$,
$f = gh$, $h = (\mathcal{O}_K : Q)$.
$\mathfrak{f}$: the conductor of $\mathcal{O}$. $N(\mathfrak{f}) = f^2$,
$\mathfrak{f}_p$: the p-part of $\mathfrak{f}$ for $p|f$.
Put $\mathfrak{g} = \prod_{p|g} \mathfrak{f}_p$, $\mathfrak{h} = \prod_{p|h} \mathfrak{f}_p$.
$\Longrightarrow \mathfrak{g} = \mathrm{Disc}(k_6/k)$
$=$ the cond. of the unique cubic resolvent ring of $\mathcal{O}_K$.

Quartic rings Q having cubic resolvent ring $\mathcal{O}$

$a_K(\mathfrak{h}) = \#\{Q \subset \mathcal{O}_K : Q$ has cubic resolvent ring $\cong \mathcal{O}\}$,
$I_k(\mathfrak{g})$: the group of frac. ideals of k, relatively prime to $\mathfrak{g}$,
$H \subset I_k(\mathfrak{g})$: the subgr. $\longleftrightarrow k_6/k$ by class field theory,
χ: the character of $I_k(\mathfrak{g})$, $\ker \chi = H$.
Then we have

(4.1) $$a_K(\mathfrak{h}) = \prod_{p|h} (1 + \chi(\mathfrak{f}_p)).$$

If $\mathrm{Disc}(k) < 0$, then the number of quartic rings Q with fixed cubic resolvent ring $\mathcal{O}$ is given by the sum

$$\sum_{g|f} \sum_{K \in \mathcal{K}_k(\mathfrak{g})} a_K(\mathfrak{h}) = \sum_{g|f} \sum_{K \in \mathcal{K}_k(\mathfrak{g})} \prod_{p|h} (1 + \chi_K(\mathfrak{f}_p)). \tag{4.2}$$

$\mathcal{K}_k(\mathfrak{g})$ is the set of isomorphism classes of quartic fields K satisfying the following conditions:

(a) The normal closure $\tilde{K}$ of K over $\mathbb{Q}$ has Galois group S_4 and contains k.

(b) The unique cubic resolvent ring of the maximal order $\mathcal{O}_K$ is isomorphic to $\mathcal{O}_0 = \mathbb{Z} + \mathfrak{g}$.

(c) K is totally real if $\mathrm{Disc}(k) > 0$.

If $\mathrm{Disc}(k) > 0$, then the sum (4.2) gives the number of such quartic rings contained in some totally real S_4-quartic fields.

If $\mathrm{Disc}(k) > 0$, denote by $\mathcal{K}_k(\mathfrak{g}\mathfrak{f}_\infty)$ the set of isomorphism classes of quartic fields K satisfying the conditions (a) and (b) above (including totally imaginary fields). If $\mathrm{Disc}(k) > 0$, then the number of quartic rings contained in some quartic fields with fixed cubic resolvent ring $\mathcal{O}$ is given by

$$(4.3) \qquad \sum_{g|f} \sum_{K \in \mathcal{K}_k(\mathfrak{g}\mathfrak{f}_\infty)} a_K(\mathfrak{h}) = \sum_{g|f} \sum_{K \in \mathcal{K}_k(\mathfrak{g}\mathfrak{f}_\infty)} \prod_{p|h} (1 + \chi_K(\mathfrak{f}_p)).$$

By class field theory and quadratic reciprocity laws over the cubic field k, we obtain the following formulae.

$$(4.4) \qquad \begin{aligned} \sum_{\mathfrak{g}|\mathfrak{f}} |\mathcal{K}_k(\mathfrak{g})| &= |\,\mathrm{Cl}_{\mathcal{O}}^{(2)}\,| - 1, \\ \sum_{\mathfrak{g}|\mathfrak{f}} |\mathcal{K}_k(\mathfrak{g}\mathfrak{f}_\infty)| &= |\,\mathrm{Cl}_{\mathcal{O},+}^{(2)}\,| - 1. \end{aligned}$$

We can rewrite the right hand sides of (4.2) and (4.3) so that we finally obtain the following formulae:

$$\text{(4.5)} \quad \sum_{g|f} \sum_{K \in \mathcal{K}_k(\mathfrak{g})} a_K(\mathfrak{h}) = \sum_{g|f} |\operatorname{Cl}^{(2)}_{R_g}|(2 - |X(\mathcal{O}, R_g)|) - 2^{\omega(f)},$$

$$\text{(4.6)} \quad \sum_{g|f} \sum_{K \in \mathcal{K}_k(\mathfrak{g}\mathfrak{f}_\infty)} a_K(\mathfrak{h}) = \sum_{g|f} |\operatorname{Cl}^{(2)}_{R_g,+}|(2 - |X_+(\mathcal{O}, R_g)|) - 2^{\omega(f)}$$

$$(\operatorname{Disc}(k) > 0).$$

Here $\omega(f) = \#\{p : p|f\}$, $R_g = \mathbb{Z} + \mathfrak{g}$.
We see

$$2^{\omega(f)} = \#\{Q \subset \mathbb{Q} \oplus k : Q \text{ has cubic resolvent ring } \cong \mathcal{O}\}.$$

We write $x = (A, B) \in L(\mathcal{O})$, $\mu(x) = 1/|\Gamma_x|$. By Proposition 3.3, $\mu(x) = 1/|\operatorname{Aut}(Q(A, B))| = 1$. It follows from Theorem 3.1, Corollary 3.2, (4.5) and (4.6) that if $\operatorname{Disc}(k) > 0$, then

(4.7)

$$\sum_{x \in \Gamma \backslash L_1(\mathcal{O})} \mu(x) = \sum_{g|f} |\operatorname{Cl}^{(2)}_{R_g}|(2 - |X(\mathcal{O}, R_g)|),$$

$$\sum_{x \in \Gamma \backslash L_1(\mathcal{O})} \mu(x) + \sum_{x \in \Gamma \backslash L_3(\mathcal{O})} \mu(x) = \sum_{g|f} |\operatorname{Cl}^{(2)}_{R_g,+}|(2 - |X_+(\mathcal{O}, R_g)|),$$

otherwise

(4.8)
$$\sum_{x \in \Gamma \backslash L_2(\mathcal{O})} \mu(x) = \sum_{g|f} |\operatorname{Cl}^{(2)}_{R_g}|(2 - |X(\mathcal{O}, R_g)|).$$

So we finally complete the proof of Theorem 1 for a non-Galois cubic field k by Proposition 2.4, (4.7) and (4.8).